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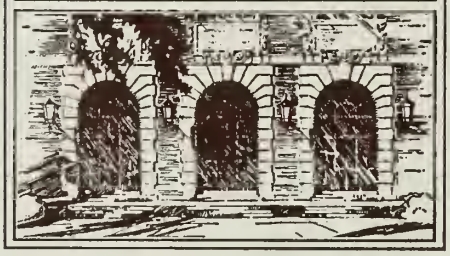

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A THEORY OF ASYNCHRONOUS CIRCUITS III

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11. Cycling in Distributive Circuits

11.1 In section 10 we began the investigation of distributive circuits and there indicated that this class of circuits would form the basis of a synthesis procedure. The remaining chapters of this paper will be devoted to this procedure. However several other properties of distributive circuits need to be established before we begin its development. These results are concerned with cycling in distributive circuits and the properties of the cycling vectors w(i). In a semi-modular circuit (see example p. 37) we saw that a vector w(i) could split into two or more vectors in passing from one equivalence class to another. Also it seems likely that the vector could pass to another having the same spanned components but with one or more of them reduced by a multiple of two. We shall see that this is not the case in a distributive circuit; that once a vector w(i) appears it cannot change in passing from one equivalence class to another. We shall begin by proving the dual result of (8:3) under the assumption of distributivity. The notation D:C[u] will be used to denote a distributive circuit C[u].

(11:1) If $\underline{a}$ and $\underline{b}$ are in D:C[u] and $t(\underline{a}) = t(\underline{b})$ then $t(\underline{a} \cap \underline{b}) = t(\underline{a})$.

Proof: Let $c = t(\underline{a} \cap \underline{b})$ then there exist sequences c, a(1), a(2),..., a and c, b(1), b(2),..., b such that $\underline{a} \cap \underline{b} + L[c,a(1),a(2),\ldots,a] = \underline{a}$ and $\underline{a} \cap \underline{b} + L[c,b(1),b(2),\ldots,b] = \underline{b}$. Assume that for the ith component $\underline{a}_i \geq \underline{b}_i$ then $(\underline{a} \cap \underline{b})_i = \underline{b}_i$ by (10:2), hence the ith component of L[c,b(1),b(2),...,b] is 0 which holds only if $c_i = b(1)_i = b(2)_i = \ldots = b_i$. On the other hand if $\underline{a}_i \leq \underline{b}_i$, the same argument would yield $c_i = a_i$, but $a_i = b_i$ by hypothesis, thus the theorem is established.

An immediate corollary to (11:1) is the following:

(11:2) Let $\underline{a}$ be in D:C[u]. Then the set of all states $\underline{b}$ such that $t(\underline{b}) = t(\underline{a})$ is a sub-distributive lattice of D:C[u] possessing a minimal element.

Proof: (8:3) and (11:1) yield closure under the lattice operations $\cup$ and $\cap$ hence the set is a sub-lattice of D:C[u]. Furthermore since D:C[u] is distributive and possesses a minimal element any sub-lattice will have the same properties (see ref. 3, p. 100).

The next result establishes the closure of an equivalence class under the lattice operation $\cap$. This result is not as strong a result as the dual of (8:4) might have been, however an example will be given later which denies the possibility of proving the stronger result.

(11:3) If $\underline{a}$ and $\underline{b}$ both lie in the same equivalence class $\underline{A}$ of a D:C[u] circuit, then $\underline{a} \cap \underline{b}$ also lies in $\underline{A}$.

Proof: Let w(i), i = 1, 2,..., k be the set of all vectors in W[A] and let m be the maximum value of $|\underline{b}_j - \underline{a}_j|$ for all components j. Consider the states $\underline{a}^* = \underline{a} + m \sum_{i=1}^{k} w(i)$ and $\underline{b}^* = \underline{b} + m \sum_{i=1}^{k} w(i)$. By (8:C) $\underline{a}^*$ lies in E[$\underline{a}$] and $\underline{b}^*$ lies in E[$\underline{b}$]. Furthermore by (8:F) $\underline{a}$ and $\underline{b}$ can differ only in their spanned components, thus $\underline{a}^*$ and $\underline{b}^*$ have been constructed so that $\underline{a} \leq \underline{b}^*$ and $\underline{b} \leq \underline{a}^*$. Let $\underline{c} = \underline{a} \cap \underline{b}$ and $\underline{c}^* = \underline{a}^* \cap \underline{b}^*$. Assuming $\underline{a}_i \leq \underline{b}_i$ we obtain $\underline{a}^*_i \leq \underline{b}^*_i$ and further by distributivity $\underline{c}_i = \underline{a}_i$ and $\underline{c}^*_i = \underline{a}^*_i$. But $\underline{c} \leq \underline{a}$ and $\underline{c}^* \leq \underline{a}^*$ hence $c_i = a_i$ and $c^*_i = a^*_i$ by the same argument as found in (11:1). Reversing the roles of $\underline{a}$ and $\underline{b}$ in this argument, we obtain: if $\underline{a}_i \geq \underline{b}_i$ then $c_i = b_i$ and $c^*_i = b^*_i$. In either case $\underline{a}^*$ lying in E[$\underline{a}$] and $\underline{b}^*$ lying in E[$\underline{b}$] implies $c_i = c^*_i$, hence $t(\underline{c}) = t(\underline{c}^*)$. However by the choice of $\underline{b}^*$ we have $\underline{c} \leq \underline{a} \leq \underline{c}^*$, thus $\underline{c} = \underline{a} \cap \underline{b}$ is in the same equivalence class as $\underline{a}$.

As before (11:3) allows us to establish the following corollary:

(11:4) The set of all states $\underline{a}$ in an equivalence class $\underline{A}$ of a D:C[u] circuit is a sub-distributive lattice of D:C[u] possessing a minimal element.

Proof: The proof is precisely the same as that in (11:2) where (8:4) and (11:3) replace (8:3) and (11:1) respectively.

11.2 Consider the following example of a distributive circuit defined by the Boolean equations:

(11:5) $$z_1' = 1$$
$$z_2' = \bar{z}_1 \vee \bar{z}_2 .$$

If the circuit is started in the initial state (0,0) there result three equivalence classes in C[u]. Two of them contain only single states; $\underline{A}$ = (0,0) and $\underline{B}$ = (0,1). The third, $\underline{C}$, consists of the remaining states in C[u]; (1,0), (1,1), (1,2), (1,3),..., and is the final class of the circuit. We note that $\underline{B} \mathfrak{F} \underline{C}$ however, the class containing the intersection of (0,1) in $\underline{B}$ and (1,0) in $\underline{C}$ is $\underline{A}$ and not $\underline{B}$ as would be predicted by the dualization of theorem (8:4).

The following lemma is needed before we are able to prove the invariance of the cycling vectors in a D:C[u] circuit.

(11:6) Let $\underline{A}$ and $\underline{B}$ be two equivalence classes in a D:C[u] circuit where $\underline{A} \mathfrak{F} \underline{B}$. If $\underline{a}$ and $\underline{b}$ are any two states in $\underline{A}$ and $\underline{B}$ respectively and if $\underline{C}$ is the equivalence class of $\underline{c} = \underline{a} \cap \underline{b}$, then W[C] = W[A].

Proof: Let w(i) be any vector in W[A] and let $\underline{a}^* = \underline{a} + w(i)$ and $\underline{b}^* = \underline{b} + w(i)$. Then $t(\underline{a}^*) = t(\underline{a})$ and by (8:D), $t(\underline{b}^*) = t(\underline{b})$. Let $\underline{c}^* = \underline{a}^* \cap \underline{b}^*$ then since the lattice is distributive $\underline{c}^* = \underline{c} + w(i)$. Using the same argument as found in (11:3) we obtain $t(\underline{c}) = t(\underline{c}^*)$, hence w(i) is a sum of vectors in W[C]. But $C \mathfrak{F} A$ and by (8:D) every vector in W[C] is a sum of vectors in W[A]. This property along with the orthogonality of the vectors in W[A] implies W[C] = W[A].

(11:A) If $\underline{A}$ and $\underline{B}$ are two equivalence classes in a D:C[u] circuit where $\underline{A} \mathfrak{F} \underline{B}$ then $W[A] \subseteq W[B]$.

Proof: We will continually make use of the argument developed in (11:1) which may be stated as: if $\underline{a} \leq \underline{b}$ and $\underline{a}_i = \underline{b}_i$ then $a_i = b_i$.

Let a be any state in A and b any state in B. Since $A \mathfrak{J} B$ there exist sequences u, a(1), a(2),..., a and a, b(1), b(2),..., b so that the C-states $\underline{a}$ = L[u, a(1), a(2),..., a] and $\underline{b}$ = L[u, a(1), a(2),..., a, b(1), b(2),... b] exist in C[u]. Let w(i) be any vector in W[A] and by (8:D) let $w(i) = \sum_{j=1}^{m} c_j w(i,j)$ where the w(i,j), j = 1, 2,..., m lie in W[B]. Let $\underline{d}$ denote the C-state $\underline{a} + \max_j |\underline{b}_j - \underline{a}_j| w(i)$ which lies in E[$\underline{a}$] and hence also in $\underline{A}$ and then define $\underline{e}$ as $\underline{e} = \underline{d} \cap \underline{b}$. Applying (11:6) to the state $\underline{e}$, letting C denote its equivalence class, we obtain W[C] = W[A]. But, in addition, if $\underline{a}_i$ is unspanned by W[A] then $\underline{d}_i = \underline{a}_i$ and since $\underline{a} \leq \underline{b}$ we would have $\underline{e}_i = \underline{a}_i$. Hence by (8:F) C = A or alternatively $\underline{e}$ lies in $\underline{A}$.

Consider now the state $\underline{e}^* = [\underline{d} + w(i)] \cap [\underline{b} + w(i,j)]$. Since $w(i,j) \leq w(i)$ we obtain $\underline{e}^* = \underline{e} + w(i,j)$ through the properties of $\cap$ in a distributive circuit. In addition $\underline{e}$ and $\underline{e} + w(i)$ being in $\underline{A}$ and $\underline{e} \leq \underline{e}^* \leq \underline{e} + w(i)$ imply $\underline{e}^*$ is also in $\underline{A}$.

Let k denote any node. If w(i) does not span node k then neither does w(i,j), hence $\underline{e}^*_k = \underline{e}_k$ or since $\underline{e} \leq \underline{e}^*$ we obtain $e^*_k = e_k$. On the other hand if w(i) spans node k then $\underline{d}$ being constructed so that $\underline{d}_k \geq \underline{b}_k$ implies $\underline{e}_k = \underline{b}_k$ or since $\underline{e} \leq \underline{b}$ we have $e_k = b_k$. However $w(i,j) \leq w(i)$ thus $[\underline{d} + w(i)]_k \geq [\underline{b} + w(i,j)]_k$. Hence $\underline{e}^*_k = [\underline{b} + w(i,j)]_k$ or since $\underline{e}^* \leq \underline{b} + w(i,j)$ and $\underline{b} + w(i,j)$ is in E[$\underline{b}$] we obtain $e^*_k = t(\underline{b} + w(i,j))_k = b_k = e_k$. Thus in either case $e_k = e^*_k$ hence $e = e^*$.

Now using (8:5b), $\underline{e} + w(i)$ covers $\underline{e}$ in E[$\underline{e}$] therefore $\underline{e}^* = \underline{e} + w(i,j)$ either equals $\underline{e}$ or $\underline{e} + w(i)$. If $w(i) \neq 0$ then some $w(i,j) \neq 0$ and $\underline{e}^* \neq \underline{e}$ hence $\underline{e}^* = \underline{e} + w(i)$ or $w(i,j) = w(i)$ which completes the proof.

11.3 Theorem (11:A) is quite important. It states that in passing from one equivalence class to another, the previous cycling vectors w(i) remain invariant and only new ones may arise. That is, once a set of nodes becomes spanned by some w(i) they will remain spanned by the same vector regardless of the behavior of the rest of the circuit. In addition, it is easily proved that there exists an invariant covering sequence on the nodes spanned by a given vector w(i), thus

after these nodes become spanned they may be studied quite apart from the remaining nodes in the circuit.

We will now introduce the definition of a causation signal. Although quite similar to the definition of a C-signal (section 10), it does not consist of a set of C-states and as a consequence will lend itself to a better development of the synthesis procedure.

(11:7) A causation signal $[\alpha,i]$ is an ordered pair of positive integers and is said to exist if and only if there exists a vector $\underline{a}$ in $C[u]$ with $\underline{a}_i = \alpha > 0$. The vector $\underline{a}$ is said to induce the causation signal $[\alpha,i]$.

We also introduce a causation relationship on causation signals which will be shown later to be a partial ordering of the set of causation signals.

(11:8) $[\alpha,i]$ causes $[\beta,j]$ ($[\alpha,i] \leq [\beta,j]$) if for every state $\underline{a}$ in $C[u]$ where $\underline{a}_i \leq \alpha - 1$ then $\underline{a}_j \leq \beta - 1$.

11.4 Intuitively we think of the causation signal $[\alpha,i]$ as the αth change that occurred at node i since the circuit was started in the initial state. The causation relationship can then be stated as: the βth change on node j cannot occur before the αth change on node i. We note that given any $[\alpha,i]$ there exists a C-signal $(\alpha-1,i)$. For let $\underline{a}$ be any state inducing $[\alpha,i]$, then since $\underline{a}_i > 0$ any covering sequence $\underline{0}$, $\underline{a}(1)$, $\underline{a}(2),\ldots,$ $\underline{a}$ contains a state $\underline{a}^*$ inducing the C-signal $(\alpha-1,i)$. Also if $[\alpha,i] \leq [\beta,j]$ then by (11:8) and (9:1) $(\alpha-1,i) \subseteq (\beta-1,j)$. Furthermore if $(\alpha-1,i) \subseteq (\beta-1,j)$ and $[\alpha,i]$ and $[\beta,j]$ exist then $[\alpha,i] \leq [\beta,j]$. With these remarks we are able to restate theorem (10:B) in terms of the causation signals. However the restatement of (10:B) could be proved directly without showing the isomorphism between the C-signals and the causation signals.

(11:B) In a D:C[u] circuit the set of non-zero join-irreducible elements is isomorphic to the set of causation signals, $[\alpha,i]$. Under this isomorphism if $[\alpha,i]$ and $[\beta,j]$ correspond to $\underline{a}$ and $\underline{b}$ respectively then $\underline{a} \leq \underline{b}$ if and only if $[\alpha,i] \leq [\beta,j]$.

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We note that the mapping developed in (10:B) now becomes: $[\alpha,i]$ corresponds to the join-irreducible element $\underline{a}$ where $\underline{a}$ is the minimal element in D:C[a] whose i^{th} component is α. We will denote by $\Sigma[u]$ the set of all causation signals $[\alpha,i]$ of a D:C[u] circuit. The remaining theorems in this chapter will be concerned with the properties of this set.

(11:C) $\Sigma[u]$ is a partially ordered set satisfying:

(11:9a) For any $[\alpha,i]$ in $\Sigma[u]$ there exists only a finite number of $[\beta,j]$ in $\Sigma[u]$ where $[\beta,j] < [\alpha,i]$.

(11:9b) For all $[\alpha,i]$ in $\Sigma[u]$ there exists an integer n such that $i \leq n$.

(11:9c) If $\Sigma[u]$ contains $[\alpha,i]$ and $\alpha > 1$ then $\Sigma[u]$ contains $[\alpha-1,i]$ and $[\alpha-1,i] \leq [\alpha,i]$.

<u>Proof</u>: We first establish that $\Sigma[u]$ is partially ordered. From (11:B) we immediately obtain $[\alpha,i] \leq [\alpha,i]$ and the transitivity of the causation relationship, $\leq$. Assume now that $[\alpha,i] \leq [\beta,j]$ and $[\beta,j] \leq [\alpha,i]$. If $\underline{a}$ and $\underline{b}$ are the join-irreducible elements corresponding to $[\alpha,i]$ and $[\beta,j]$ respectively, then we have $\underline{a} \leq \underline{b}$ and $\underline{b} \leq \underline{a}$ hence $\underline{a} = \underline{b}$. Now let $\underline{c}$ be any state covered by $\underline{a}$; such a state must exist since $\underline{a} \neq 0$. Clearly $\underline{c}_i < \underline{a}_i$ since $\underline{a}$ is the minimal state in D:C[u] having i^{th} component α. Also by the same argument $\underline{c}_j < \underline{a}_j$. But $\underline{c}$ differs from $\underline{a}$ in only one component hence $i = j$ and $\alpha = \beta$. Therefore $\Sigma[u]$ is partially ordered.

Next condition (11:9a) is easily established, for let $\underline{a}$ correspond to $[\alpha,i]$, then since all components of $\underline{a}$ are finite, there is only a finite number of states which $\underline{a}$ may follow, hence only a finite number of join-irreducible elements. Condition (11:9b) is trivial since the number of components is finite.

To establish (11:9c) we begin by taking any state $\underline{d}$ inducing $[\alpha,i]$ and form a covering sequence $\underline{0}$, $\underline{d}(1)$, $\underline{d}(2)$,..., $\underline{d}$. Since $\alpha > 1$ we can find some state $\underline{a}^*$ inducing $[\alpha-1,i]$. Now let $\underline{a}$ and $\underline{b}$ be the join-irreducible elements corresponding to $[\alpha-1,i]$ and $[\alpha,i]$ respectively and consider the state $\underline{c} = \underline{a} \cap \underline{b}$. We must have

$\underline{c}_i = \alpha - 1$ by distributivity. Furthermore since $\underline{a} \leq \underline{c}$ ($\underline{a}$ being the minimal state satisfying $\underline{a}_i = \alpha - 1$), we have $\underline{a} \leq \underline{a} \cap \underline{b}$ hence $\underline{a} \leq \underline{b}$. Finally by (11:B) we obtain the desired result, $[\alpha\text{-}1,i] \leq [\alpha,i]$.

11.5 We note that theorem (11:C) is a characterization of the lattice of a circuit by the set of causation signals. Indeed, if the circuit does not cycle then this characterization by conditions (11:9) will be taken later to be the basis for the synthesis procedure. However if the circuit cycles, then since there are isomorphisms within the lattice, we would expect to also find a cyclic structure in the set $\Sigma[u]$. The remaining development in this chapter will be devoted to describing this cyclic structure. In the following theorems we shall use the notation w to denote an arbitrary vector rather than the usual notation w(i).

(11:10) Let w be in some W[A] in a D:C[u] circuit. Then there exists an equivalence class B such that w is in W[B] and further, if C is any equivalence class such that w is in W[C] then $B \mathrel{\underline{\supset}} C$.

<u>Proof</u>: Let $\underline{B}(1), \underline{B}(2), \ldots, \underline{B}(j)$ be all the equivalence classes such that w is in W[B(i)] and by (11:4) let $\underline{b}(1), \underline{b}(2), \ldots, \underline{b}(j)$ be the corresponding minimal states of the sub-lattices. Let $\underline{A}$ denote the equivalence class of the state $\underline{a} = \underline{b}(1) \cap \underline{b}(2) \cap \ldots \cap \underline{b}(j)$. Then by the properties of $\cap$ in a distributive lattice we obtain that $\underline{b}(1) + w \cap \underline{b}(2) + w \cap \ldots \cap \underline{b}(j) + w = \underline{a} + w$. Furthermore $t(\underline{a}+w) = t(\underline{a})$ by the argument used in the proof of (11:3); thus w is in W[A] hence $\underline{A}$ must be one of the $\underline{B}(i)$, say $\underline{B}(r)$. But $\underline{b}(r)$ is minimal in $\underline{B}(r)$ therefore we have $\underline{b}(r) \leq \underline{a}$, hence $\underline{b}(r) = \underline{b}(1) \cap \underline{b}(2) \cap \ldots \cap \underline{b}(j)$. Therefore every $\underline{b}(k)$ satisfies $\underline{b}(r) \leq \underline{b}(k)$ hence $\underline{B}(r) \mathrel{\underline{\supset}} \underline{B}(k)$ for all k, thus completing the proof.

Theorem (11:10) states that given a vector w there is a minimal equivalence class A such that w is in W[A]. We will say that A is minimal with respect to the given vector w. The following example shows that A may be minimal with respect to two or more cycling vectors. The circuit is defined by the Boolean equations:

(11:11) $z_1' = \bar{z}_1$

$z_2' = \bar{z}_2$.

This circuit when started in (0,0) has only one equivalence class which is minimal with respect to both (2,0) and (0,2), the two cycling vectors. In order to more conveniently state the results which are to follow, we shall use the notation $[\alpha,i] \sim \underline{a}$ to indicate $\underline{a}$ is the join-irreducible element corresponding to the causation signal $[\alpha,i]$.

(11:12) Let w be in some W[A] in a D:C[u] circuit where $w_i \neq 0$. Then there exist join-irreducible elements $\underline{a} \sim [\alpha,i]$ and $\underline{b} \sim [\beta,i]$ such that $\underline{a} < \underline{b}$, $t(\underline{a}) = t(\underline{b})$ and $\underline{b} = \underline{a} + rw$ where r is some positive integer.

<u>Proof</u>: Since $w_i \neq 0$ then $\underline{j}(k) \sim [k,i]$ exist for all $k \geq 1$. Furthermore there is only a finite number of possible states $t(\underline{j}(k))$, hence integers p and q exist for which $t(\underline{j}(p)) = t(\underline{j}(q))$ and $p > q$. Therefore $\underline{j}(p)$ may be written as $\underline{j}(p) = \underline{j}(q) + \sum_i a_i\, w(i)$ where the w(i) are cycling vectors of the circuit. Assume $a_1\, w(1) \neq 0$, then since the w(i) are orthogonal we may write $\underline{j}(p) = [\underline{j}(q) + a_1\, w(1)] \cup [\underline{j}(q) + \sum_{i \neq 1} a_i\, w(i)]$. But $\underline{j}(p)$ is join-irreducible hence through the orthogonality of the w(i) we obtain $\sum_{i \neq 1} a_i\, w(i)$ must be 0. Moreover since $\underline{j}(p)_i > \underline{j}(q)_i$ we have both w(1) and w spanning node i; hence through the invariance of the w(i) (theorem (11:A)), we obtain w = w(1).

(11:13) Let $\underline{j} \sim [\alpha,i]$ and lie in $\underline{A}$. Let w be in W[A] where $w_i \neq 0$. Then A is minimal with respect to w and if $\underline{a} \sim [\beta,i]$ where $\beta > \alpha$ then $\underline{a}$ also lies in A.

<u>Proof</u>: Let $\underline{B}$ be minimal with respect to w, then by (11:10) $\underline{B} \mathfrak{J} \underline{A}$. Let $\underline{b}$ be the minimal element of $\underline{B}$ then since w is in W[B], we may form the state $\underline{c} = \underline{b} + |\underline{b}_i - \underline{j}_i|w$. This state lies in $\underline{B}$, however since $w_i \neq 0$ its ith component is greater than the ith component of $\underline{j}$. Thus $\underline{j} \leq \underline{c}$ and $\underline{j}$ being in $\underline{A}$ implies $\underline{A} \mathfrak{J} \underline{B}$, hence $\underline{A} = \underline{B}$.

Now let $\underline{a} \sim [\beta, i]$ then if $\beta > \alpha$, we have $\underline{j} < \underline{a}$. The vector $\underline{j} + |\underline{a}_i - \underline{j}_i| w$ is in $\underline{A}$ and its ith component is $\geq \underline{a}_i$. Hence $\underline{j} < \underline{a} \leq \underline{j} + |\underline{a}_i - \underline{j}_i| w$ or $\underline{a}$ is also in $\underline{A}$.

11.6 Theorem (11:10) has established the existence of a minimal equivalence class corresponding to a given vector w. Furthermore (11:12) states that if $w_i \neq 0$, there exist join-irreducible elements $\underline{a} \sim [\alpha, i]$ and $\underline{a}^* = a + rw \sim [\beta, i]$ where r is a positive integer and both $\underline{a}$ and $\underline{a}^*$ lie in some equivalence class $\underline{B}$. Finally (11:13) states that $\underline{B}$ must be the equivalence class $\underline{A}$. From these remarks the following corollary is immediately established.

(11:14) Let $\underline{A}$ be minimal with respect to w. Then if $w_i \neq 0$ there exists a join-irreducible element $\underline{a}$ lying in $\underline{A}$ and corresponding to some $[\alpha, i]$.

(11:D) Let $\underline{j} \sim [\alpha, i]$ and $\underline{j} + aw \sim [\gamma, i]$ where $\gamma > \alpha$ and $\underline{j} + aw$ lies in $E[\underline{j}]$. Then $\underline{j} + w \sim [\beta, i]$ where $\beta = \alpha + w_i$.

<u>Proof</u>: Assume that $\underline{j} + w$ is not join-irreducible, then there exist states $\underline{a}$ and $\underline{b}$ covered by $\underline{j} + w$ such that $\underline{j} + w = \underline{a} \cup \underline{b}$. Since $\underline{j}$ is join-irreducible and $\underline{j} < \underline{j} + w$, then either $\underline{j} \leq \underline{a}$ or $\underline{j} \leq \underline{b}$ (either $\underline{j}_i \leq \underline{a}_i$ or $\underline{j}_i \leq \underline{b}_i$). We will assume $\underline{j} \leq \underline{a}$ and will show that also $\underline{j} \leq \underline{b}$.

Since $\underline{b}$ is covered by $\underline{j} + w$ then for some single component k we have $(\underline{j} + w)_k = \underline{b}_k + 1$. Now either w spans node k or not. If k is spanned by w, then since $w_k \geq 1$ we would have $\underline{j}_k \leq \underline{b}_k$, hence $\underline{j} \leq \underline{b}$. Thus, we assume k is not spanned by w, then $\underline{j}_k = (\underline{j} + w)_k = \underline{b}_k + 1$.

Consider now the vector $\underline{b} \cap \underline{j}$. By the development in the last paragraph we have $(\underline{b} \cap \underline{j})_k + 1 = \underline{b}_k + 1 = \underline{j}_k$. For all other components m we have $(\underline{b} \cap \underline{j})_m = \underline{j}_m$, hence $(\underline{b} \cap \underline{j})$ is covered by $\underline{j}$. But $\underline{j}$ is the minimal state having $\underline{j}_i = \alpha$ and since $(\underline{b} \cap \underline{j})_i = \underline{j}_i = \alpha$ (i is spanned by w and hence $i \neq k$), we have a contradiction. Thus w spans node k and both $\underline{j} \leq \underline{a} < \underline{j} + w$ and $\underline{j} \leq \underline{b} \leq \underline{j} + w$ hold. Finally $\underline{a}$ and $\underline{b}$ must therefore both lie in the same equivalence class as $\underline{j}$.

We now form the vector $\underline{c} = (\underline{a} + rw) \cup (\underline{b} + rw)$ where r is any positive integer. Then by the properties of $\cup$ and the previous development we obtain $\underline{c} = \underline{a} \cup \underline{b} + rw = \underline{j} + (r + 1)w$. Since $\underline{c}$ is not join-irreducible then neither is $\underline{j} + (r + 1)w$ for all positive r, thus contradicting the hypothesis. Hence $\underline{j} + w$ must be join-irreducible.

Now let $\underline{b}$ be covered by $\underline{j} + w$. Then since $\underline{j} < \underline{b} < \underline{j} + w$ we have $\underline{b}$ in the same equivalence class as $\underline{j}$, hence $\underline{b} + (a - 1)w$ may be formed. But $\underline{j} + aw \sim [\beta, i]$ and since $\underline{j} + aw$ covers $\underline{b} + (a - 1)w$, we have $\underline{b} + (a - 1)w$ differing from $\underline{j} + aw$ in the ith component. Thus $\underline{b}$ differs from $\underline{j} + w$ only in the ith component and therefore $\underline{j} + w$ must correspond to $[\beta, i]$ where $\beta = \alpha + w_i$ which completes the proof.

We note that theorem (11:D) states that in theorem (11:12) the integer is equal to 1. Also it seems at first glance that a stronger result than (11:D) could have been proved. That being: if $\underline{A}$ is minimal with respect to w where $w_i \neq 0$ and further if $\underline{j}$ is in $\underline{A}$ and corresponds to some $[\alpha, i]$, then $\underline{j} + w$ is join-irreducible and corresponds to $[\alpha + w_i, i]$. However the following example shows that this is not the case. The circuit is described by the Boolean equations:

(11:15)
$$z_1' = \bar{z}_2 \vee z_1 z_3$$
$$z_2' = z_1 \vee z_2 \bar{z}_3$$
$$z_3' = \bar{z}_1 z_2$$

Starting this circuit in the state (1,0,0) gives rise to only one equivalence class consisting of all possible states. Figure 4 shows the I-states, C-states, and causation signals for this circuit.

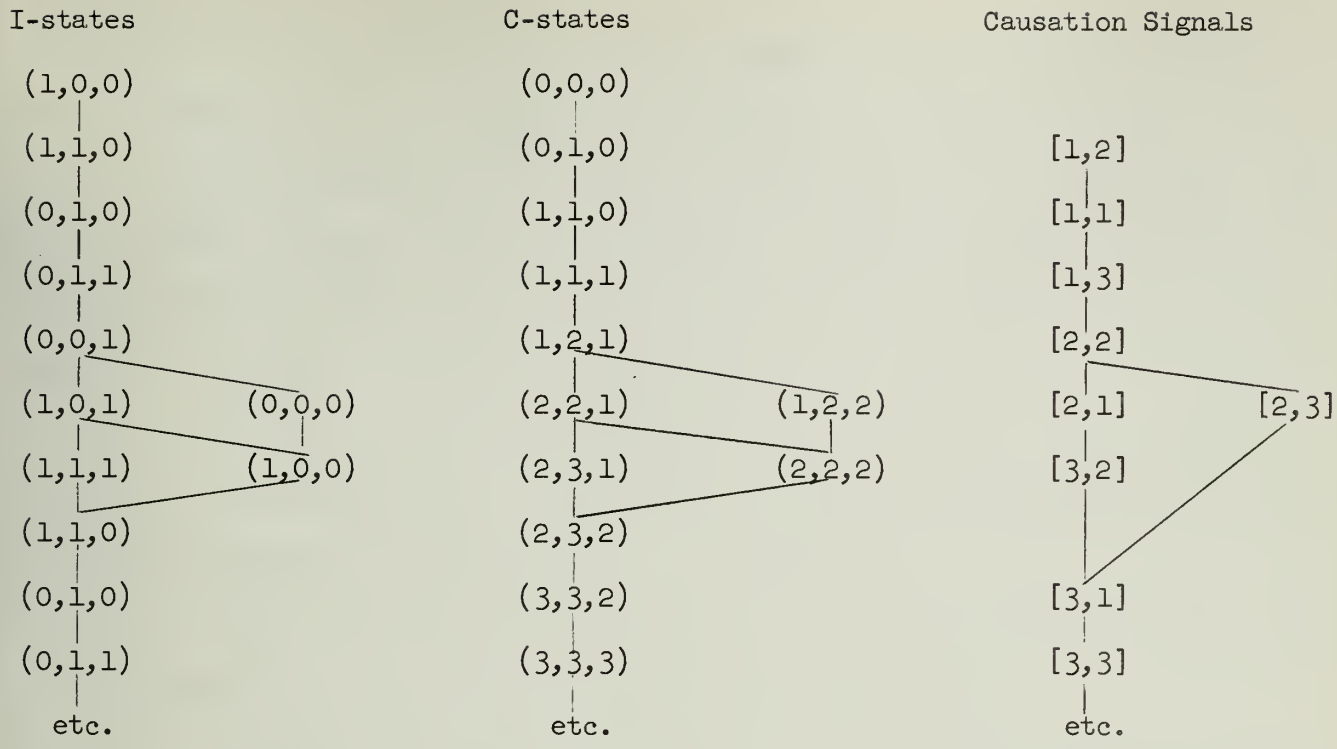

Figure 4

We note that the join-irreducible element (0,1,0) corresponds to [1,2], however the state (0,1,0) + w = (0,1,0) + (2,2,2) = (2,3,2) is not join-irreducible. As a consequence of this example we must define the following concept.

(11:16) Let $\underline{a} \sim [\alpha,i]$ then $[\alpha,i]$ is said to be cyclicly spanned by w if and only if $w_i \neq 0$ and $\underline{a} + w$ is in $E[\underline{a}]$ and corresponds to $[\alpha + w_i, i]$.

11.7 We note that theorem (11:12) along with (11:D) proves the existence of cyclicly spanned causation signals. Furthermore we can begin to see the relationship developing between ismorophisms in the set of causation signals and isomorphisms in the lattice. This relationship will be developed in the final theorem of this chapter however before we may state this result two lemmas must firstly be established.

(11:17) If $[\alpha,i]$ is cyclicly spanned by w then $[\beta,i]$ is cyclicly spanned by w for all $\beta > \alpha$.

Proof: Let $\underline{b} \sim [\beta,i]$ for any $\beta > \alpha$, and let $\underline{a} \sim [\alpha,i]$. Since $\underline{a} < \underline{b}$ then $\underline{b} + w$ is in $E[\underline{b}]$; furthermore $w_i \neq 0$ hence there is some integer $r \geq 1$ such that $\underline{a} < \underline{b} < \underline{a} + rw \leq \underline{b} + w$. Clearly if $\underline{a} + rw = \underline{b} + w$ the result is immediate therefore we may assume that $\underline{a} + rw < \underline{b} + w$. Now let $\underline{c}$ be any state covered by $\underline{b} + w$ and

consider the state $\underline{d} = \underline{c} \cap \underline{a} + rw$. We have assumed that $\underline{a} + rw < \underline{b} + w$ thus we must also have $\underline{a} + (r - 1) w < \underline{b}$. In view of the facts that $\underline{a} + (r - 1) w \sim [\alpha + (r - 1) w_i, i]$ and $\underline{b} \sim [\beta, i]$ we have in addition $(\underline{a} + (r - 1) w)_i < \underline{b}_i$. But since $\underline{c}$ differs from $\underline{b} + w$ in only one component we have $(\underline{a} + rw)_i \leq \underline{c}_i$ hence the ith component of d is $(\underline{a} + rw)_i$. Since $\underline{a} + rw \sim [\alpha + rw_i, i]$ we have $\underline{a} + rw \leq \underline{d}$ hence $\underline{a} + rw \leq \underline{c} < \underline{b} + w$ since $\underline{d} = \underline{c} \cap \underline{a} + rw$.

However we have $r \geq 1$ thus an isomorphic image of $\underline{c}$, $(\underline{c} - w)$ exists which is covered by $\underline{b}$. Therefore if $\underline{b} + w$ is not join-irreducible then neither is $\underline{b}$ since $\underline{c}$ was chosen arbitrarily, thus the theorem is established.

(11:18) Let $[\alpha, i]$ be cyclicly spanned by w. Then for any $w_j \neq 0$ there exists $[\beta, j]$ cyclicly spanned by w satisfying $[\alpha, i] \leq [\beta, j]$.

(11:18a) Furthermore if $[\alpha, i] < [\beta, j]$ for any $w_j \neq 0$ then $[\beta, j]$ is cyclicly spanned by w provided the circuit is binary.

<u>Proof:</u> Let $\underline{a} \sim [\alpha, i]$. We have by (11:12) and (11:D) that for every $w_j \neq 0$ some $\underline{b} \sim [\gamma, j]$ exists such that $[\gamma, j]$ is cyclicly spanned by w. The vector $\underline{c} = \underline{b} + |\underline{a}_i - \underline{b}_i| w$ has the property that $\underline{c}_i \geq \underline{a}_i$ and further it corresponds to $[\gamma + |\underline{a}_i - \underline{b}_i| w_j, j]$. But since $\underline{c}_i \geq \underline{a}_i$ we have $\underline{c} \geq \underline{a}$ hence if we take $\beta = \gamma + |\underline{a}_i - \underline{b}_i| w_j$ the first part of the theorem is established.

To establish (11:18a) we let $\underline{b} \sim [\beta, j]$, then since $\underline{a} < \underline{b}$ we have $\underline{b} + w$ exists. Assume that $\underline{b} + w$ is not join-irreducible, letting $\underline{b} + w = \underline{c} \cup \underline{d}$ where both $\underline{c}$ and $\underline{d}$ are covered by $\underline{b} + w$. Also let m denote the greatest integer in $|\underline{b}_i - \underline{a}_i| / w_i$ and form the state $\underline{a} + (m + 1) w = \underline{a}^*$. This state has its ith component $\leq (\underline{b} + w)_i$ and since it corresponds to $[\alpha + (m + 1) w_i, i]$ we have $\underline{a}^* \leq \underline{b} + w$. Moreover $\underline{a}^* \neq \underline{b} + w$ since $\underline{b} + w$ was assumed to be non join-irreducible thus $\underline{a}^* < \underline{b} + w$.

Since $\underline{c}$ and $\underline{d}$ are both covered by $\underline{b} + w$ and since $\underline{c} \neq \underline{d}$ it follows that at least one of them, say $\underline{c}$, satisfies $\underline{c}_i = (\underline{b} + w)_i$ hence $\underline{a}^* \leq \underline{c}$. If now $\underline{a}^*$ were

$\leq \underline{d}$ then since $\underline{a}^* - w = \underline{a} + mw$ exists we would have $\underline{c} - w$ and $\underline{d} - w$ both existing and covered by $\underline{b}$ contradicting $\underline{b}$ being join-irreducible. It follows then that $\underline{a}^* \not\leq \underline{d}$ hence by (10:8) we obtain $\underline{d}_i \leq \underline{a}^*_i - 1$. Also since $\underline{a}^*_i \leq \underline{b}_i + w_i$, we have $\underline{d}_i + 1 \leq \underline{b}_i + \underline{w}_i$. But $\underline{b} + w$ differs from $\underline{d}$ in only one component, hence $\underline{d}_i = \underline{b}_i + w_i - 1 = \underline{a}^*_i - 1$ and $\underline{d}_j = \underline{b}_j + w_j \geq \underline{a}_j$ for $j \neq i$.

Now let $\underline{e}^*$ be covered by $\underline{a}^*$. Since $\underline{a}^*$ is minimal with respect to the value of the ith component we must have $\underline{e}^*_i = \underline{a}^*_i - 1$, hence $\underline{e}^* \mathfrak{J} \underline{d}$. Furthermore $\underline{e}^* \mathfrak{J} \underline{c}$ since $\underline{a}^* \leq \underline{c}$. Similarly if $\underline{e}$ is covered by $\underline{a}$ then $\underline{e}_i = \underline{a}_i - 1$. But further since $\underline{a} = \underline{a}^* \bmod 2$ (a consequence of the components of w being even), $\underline{e}^* = \underline{e} \bmod 2$ and in a binary circuit $\underline{e}$ and $\underline{e}^*$ would be isomorphic states. Thus there would exist states following $\underline{e}$ which were isomorphic to $\underline{c}$ and $\underline{d}$ implying $\underline{b}$ was not join-irreducible. Hence $\underline{b} + w$ must be join-irreducible and further since $\underline{b} \sim [\beta, j]$, $\underline{b} + w$ corresponds to some $[\gamma, j]$.

Theorem (11:18) is rather interesting as (11:18a) pertains only to binary circuits. An example which illustrates the necessity of this restriction is shown in Figure 5. Node 1 is assumed to possess three values, the others only two.

In this example $[1,1] \sim (1,0,0)$ is cyclicly spanned by $w = (2,2,2)$, however $[1,2] \sim (1,1,0)$ is not, even though $[1,1] < [1,2]$. The fact that functions (not Boolean) exist which will produce this lattice of states is left for later results. However we point out now that in the C-state diagram, isomorphic states have the same excited and equilibrium nodes. Furthermore the values of z_i' are uniquely determined for these isomorphic states. We are now in a position to characterize the isomorphisms on the set of causation signals.

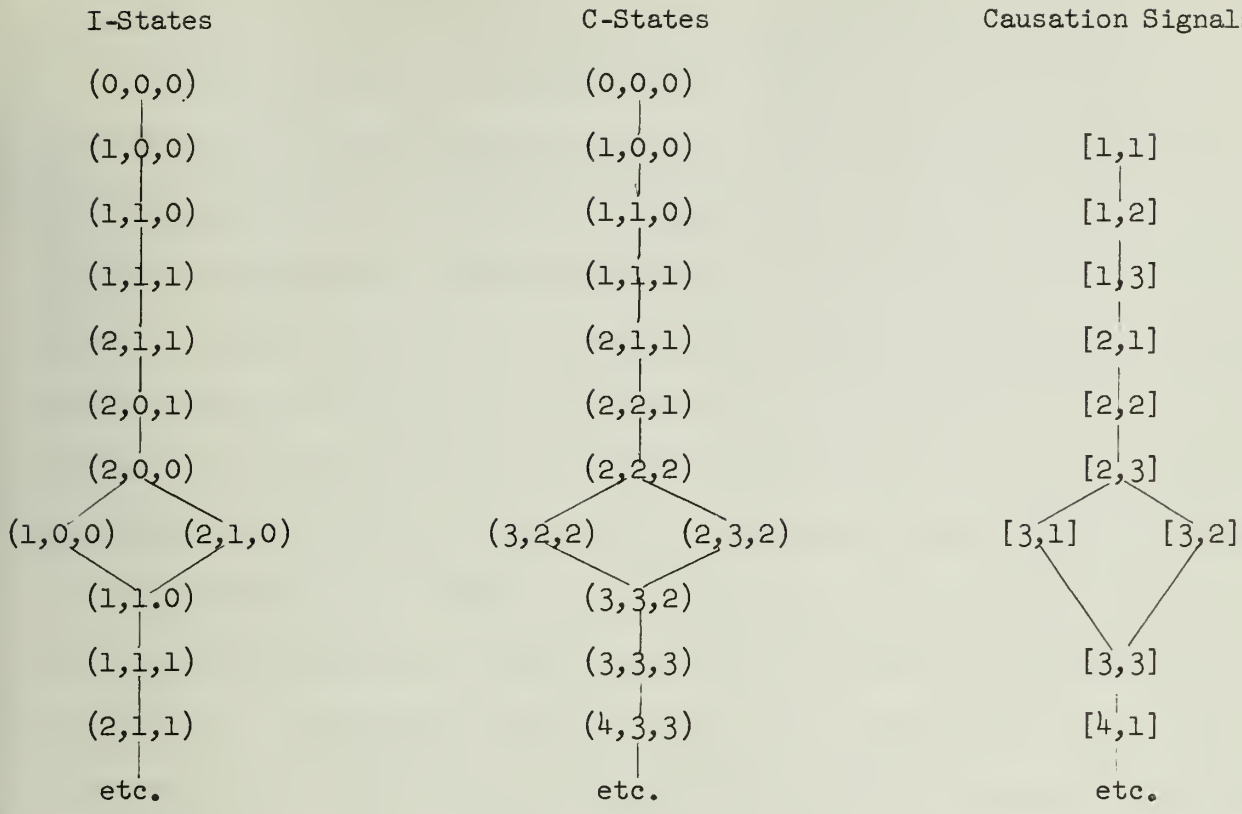

Figure 5

(11:E) Let $i(1), i(2), \ldots, i(k)$ be the set of all components for which $[\gamma, i(j)]$ exists in $\Sigma[a]$ for all γ. Then there exists a set of k signal values $\alpha(1), \alpha(2), \ldots, \alpha(k)$ and a set of k integers $r(1), r(2), \ldots, r(k)$ such that for every component $i(s)$ and $\alpha \geq \alpha(s)$: $[\alpha, i(s)] \leq [\beta, i(t)]$ if and only if $[\alpha + r(s), i(s)] \leq [\beta + r(t), i(t)]$. Furthermore there exists $[\gamma, i(s)]$ such that $[\beta, i(t)] \leq [\gamma, i(s)]$.

Proof: The set of nodes $i(1), i(2), \ldots, i(k)$ is the set of all spanned nodes since only for these nodes does $[\gamma, i(j)]$ exist for all positive γ. By [11:A] the $w(i)$ are invariant hence the set $r(1), r(2), \ldots, r(k)$ where $r(s)$ is the $i(s)^{th}$ component of that $w(i)$ which spans the $i(s)^{th}$ node is well defined. To find the integers $\alpha(s)$ we begin by considering the sequence $[1, i(s)] < [2, i(s)] < \ldots < [m, i(s)] < \ldots$. By (11:12) and (11:D) some member of this

sequence, say $[\alpha, i(s)]$, is cyclicly spanned by that w having $w_{i(s)} \neq 0$.

We wish to choose $[\alpha, i(s)]$ such that: if $[\beta,j]$ is any causation signal satisfying $[\alpha, i(s)] \leq [\beta,j]$ then firstly $w_j \neq 0$ and secondly $[\beta,j]$ is cyclicly spanned by w. We note that by (11:18) such a $[\beta,j]$ exists when $w_j \neq 0$. Thus in view of (11:17) we can choose α large enough so that if $[\alpha, i(s)] \leq [\gamma,m]$ and $w_m \neq 0$ then $[\gamma,m]$ is cyclicly spanned by w. However it may happen that after $[\alpha, i(s)]$ has been so chosen some $[\gamma,m]$ exists such that $[\alpha, i(s)] \leq [\gamma,m]$ and $w_m = 0$. Let $\underline{a} \sim [\alpha, i(s)]$ and $\underline{b} \sim [\gamma,m]$ and consider the state $\underline{a}^* = \underline{a} + |\underline{b}_{i(s)} - \underline{a}_{i(s)}|w$ which corresponds to $[\alpha + |\underline{b}_{i(s)} - \underline{a}_{i(s)}| \, w_{i(s)}, i(s)]$ as a result of $[\alpha, i(s)]$ being cyclicly spanned by w. Since $w_{i(s)} \neq 0$ this state has its $i(s)^{th}$ component $\geq \underline{b}_{i(s)}$ and hence $\underline{a}^* \not\leq \underline{b}$. But $\underline{a}_m \leq \underline{b}_m - 1$ by (10:8) and $w_m = 0$, thus $\underline{a}^*_m = \underline{a}_m$ hence $\underline{a}^* \not\geq \underline{b}$. Furthermore let $\underline{a}^* \leq \underline{c} \sim [\delta, i(s)]$, then since $\underline{c} \leq \underline{a}^* + |\underline{c}_{i(s)} - \underline{a}_{i(s)}|w$ and since $w_m = 0$ we have $\underline{a}^*_m = \underline{c}_m$. Consequently we must have $\underline{c} \not\geq \underline{b}$ and $\underline{c} \not\leq \underline{b}$. Hence $[\alpha, i(s)]$ can be chosen large enough so that if $[\alpha, i(s)] \leq [\beta,j]$ then $w_j \neq 0$.

Now let $\underline{a} \sim [\alpha, i(s)]$ and $\underline{b} \sim [\beta, i(t)]$ where both $[\alpha, i(s)]$ and $[\beta, i(t)]$ are cyclicly spanned by w and $[\alpha, i(s)] \leq [\beta, i(t)]$. Then we have $\underline{a} \leq \underline{b}$ hence $\underline{a} + w \leq \underline{b} + w$. But $\underline{a} + w \sim [\alpha + w_{i(s)}, i(s)] = [\alpha + r(s), i(s)]$ and $\underline{b} + w \sim [\beta + w_{i(t)}, i(t)] = [\beta + r(t), i(t)]$ hence $[\alpha + r(s), i(s)] \leq [\beta + r(t), i(t)]$. Similarly if $[\alpha, i(s)] \not\leq [\beta, i(t)]$ then $[\alpha + r(s), i(s)] \not\leq [\beta + r(t), i(t)]$. Finally applying (11:18) to $[\beta, i(t)]$ and the $i(s)^{th}$ component of w we obtain that there exists a $[\gamma, i(s)]$ satisfying $[\beta, i(t)] \leq [\gamma, i(s)]$ thus completing the proof.

11.8 There are several interesting facts which may now be observed. The first of these concerns the structure of $\Sigma[u]$. From (11:E) we see that the node indices $i(1), i(2), \ldots, i(k)$ may be split into mutually exclusive and jointly exhaustive classes by the relation: $i(s)\ R\ i(t)$ if and only if $[\alpha, i(s)] \leq [\beta, i(t)]$

where $\alpha > \alpha(s)$. Each of these classes turned out to be just those nodes spanned by a given cycling vector w. Furthermore the relation implies the existence of mutually exclusive classes of causation signals which may be formed as follows. Let $i(s_1), i(s_2), \ldots, i(s_t)$ be a set of node indices which was determined by the R relationship just defined. Then firstly define $\{[\alpha,i]\}$ as the set of all causation signals $[\beta,j]$ satisfying $[\alpha,i] \leq [\beta,j]$. The desired set corresponding to the nodes $i(s_1), i(s_2), \ldots, i(s_t)$ is then given by $\{[\alpha(s_1), i(s_1)]\} \cup \{[\alpha(s_2), i(s_2)]\} \cup \ldots \cup \{[\alpha(s_t), i(s_t)]\}$. Hence we see that $\Sigma[u]$ can be considered as consisting of two parts. The first of these is a finite structure and the second consists of a finite number of mutually exclusive infinite sets to be called "cycling sets". For each cycling set there is an isomorphism carrying $\{[\alpha(s_1), i(s_1)]\} \cup \{[\alpha(s_2), i(s_2)]\} \cup \ldots \cup \{[\alpha(s_t), i(s_t)]\}$ onto $\{[\alpha(s_1) + r(s_1), i(s_1)]\} \cup \{[\alpha(s_2) + r(s_2), i(s_2)]\} \cup \ldots \cup \{[\alpha(s_t) + r(s_t), i(s)]\}$ where $[\beta, i(s_m)]$ is mapped onto $[\beta + r(s_m), i(s_m)]$. Finally within each of these cycling sets we have that if $[\alpha,i] \leq [\beta,j]$ then there exists $[\gamma,i]$ such that $[\beta,j] \leq [\gamma,i]$. This latter property has the following rather interesting interpretation. We note that firstly α changes must occur at node i before β changes occur at node j. Consider now the case where the jth node is prevented from undergoing β changes. In this case the ith node of the circuit would not get beyond γ changes since $[\beta,j] \leq [\gamma,i]$. Thus we are able to define a coupling coefficient between two nodes as follows: for each $[\beta,j]$ in a cycling set let $[\alpha,i]$ be chosen so that $[\alpha,i] \leq [\beta,j]$ and $[\alpha + 1, i] \not\leq [\beta,j]$ provided of course $[\alpha,i]$ is in the same cycling set. Then let $[\gamma,i]$ be the smallest such signal satisfying $[\beta,j] \leq [\gamma,i]$. Let $c([\beta,j]) = |\gamma - \alpha|$ and finally define the coupling coefficient, c, as the maximum value of $c([\beta,j])$. It would seem at first glance that this coupling coefficient might possibly be a simple function of the cycling vector which spanned nodes i and j, however the following example shows that this

coefficient may be made as large as we please. The change chart for this circuit is shown in Figure 6.

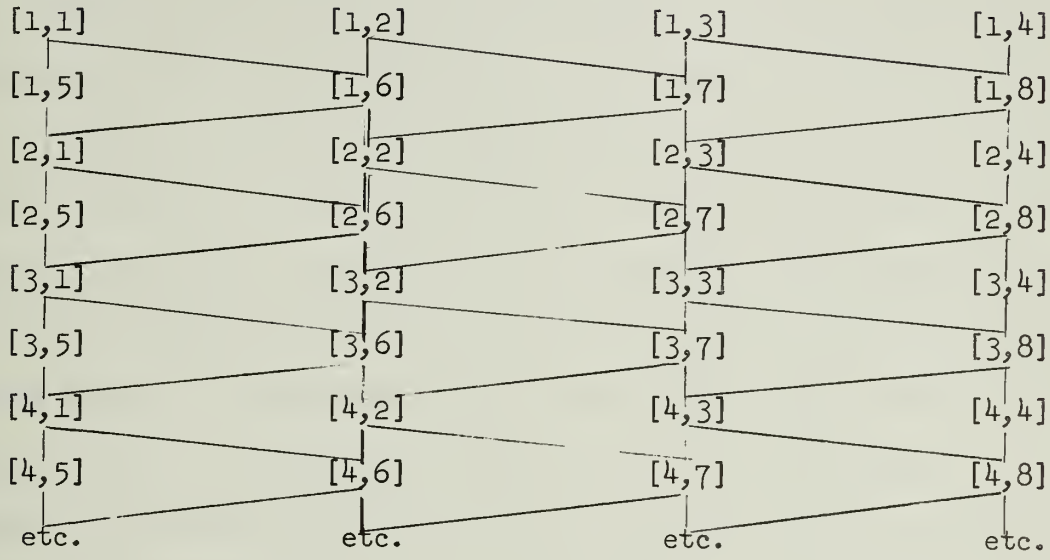

Figure 6

The circuit which will produce this change chart is shown in Figure 7. The initial state is (0,0,0,0,1,1,1,1) for nodes 1 through 8 respectively. The cycling vector, (2,2,2,2,2,2,2,2), spans all nodes. The "del" element is a "delay" element and has the Boolean equation $z_0' = z_i$ where z_i is the input and z_0 the output of the element.

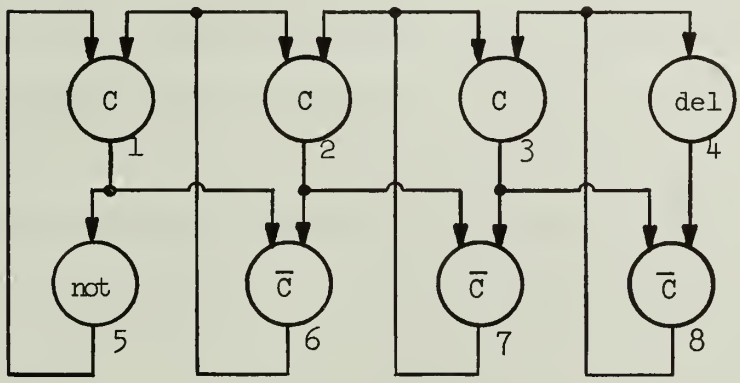

Figure 7

We note that the coupling coefficient between nodes one and eight is 5, that is, if node eight is prevented from changing, node one could change its state 5 times before the effect of node eight is felt.

11.9 Theorem (11:E) completes the development of the properties of distributive circuits. The next section will be a development of a synthesis procedure which has at its basis, a partial ordering of the changes occuring at the nodes. This partial ordering, called a change chart, is assumed to satisfy the conditions of both (11:C) and (11:E). From this chart a procedure is developed which finally obtains a set of functions that will give rise to a distributive circuit whose set of causation signals corresponds to the original change chart. The use of the change chart rather than some other description is quite advantageous as it deals with changes at individual nodes rather than states and hence is a more natural description of the circuit's behavior. Furthermore, the change chart contains fewer elements than a state diagram when parallel changes occur (approximately the logarithm of the number of possible states).

12. A General Synthesis Procedure

12.1 In this chapter we will develop a general synthesis procedure and save for a later work the applications of this procedure to the design of particular circuits. Consequently this chapter will contain a strictly mathematical development of this procedure, but whenever possible, references will be made to circuit design and analysis. The development will consist firstly of showing the equivalence between a class of distributive lattices and a class of partially ordered sets. This class of lattices will contain all the lattices which are generated by distributive circuits D:C[u]. Secondly the class of partially ordered sets, to be called change charts, will satisfy the

conditions of theorem (11:C). In addition, if a given change chart also satisfies the conditions of theorem (11:E), then by propitiously adding new nodes to the change chart while keeping the same orderings on the subset of the original nodes, a set of functions can be found which will give rise to a distributive circuit. This circuit will then have a set of causation signals equal to that of the modified change chart. We begin by defining a change chart Σ.

(12:1) A change chart Σ is any partially ordered ($\leq$) set consisting of not necessarily all distinct positive integer pairs $[\alpha,i]$ such that:

(12:1a) For all $[\alpha,i]$ in Σ there exists an integer n such that $i \leq n$ and there is only a finite number of $[\beta,j] < [\alpha,i]$.

(12:1b) If Σ contains $[\alpha,i]$ and $\alpha > 1$ then Σ contains $[\alpha-1,i]$ and $[\alpha-1,i] < [\alpha,i]$.

We will think of Σ as describing the partial ordering of the changes occurring at the nodes of an asynchronous circuit. The second index of the signal will be thought of as denoting the node in the circuit and the first index the number of the change occurring at that node since it was started in its initial state. For example [3,6] is thought of as the 3rd change at node 6 and the relation $[3,6] < [5,1]$ states that before the 5th change can occur at node 1 the 3rd change of node 6 must have occurred. If we now examine conditions (12:1) we see that (12:1a) states that the number of nodes in the circuit is finite and that to initiate a given change we need only a finite number of previous changes. This is quite analogous to the fact that a semi-modular circuit reaches a final equivalence class after a finite number of changes. The second condition of (12:1) also has a physical interpretation. It states the obvious fact that the changes on the

same node must be totally ordered and monotonely increasing. For each change chart Σ we are able to construct a set of n-dimensionsl vectors as follows:

(12:2) Λ is a set of n-dimensionsl vectors $\underline{a}$ whose components $\underline{a}_i$ are non-negative integers chosen so that:

(12:2a) If $\underline{a}_i > 0$ then $[\underline{a}_i, i]$ is in Σ and

(12:2b) If $[\beta, j] \leq [\underline{a}_i, i]$ for any $[\beta, j]$ in Σ then $\beta \leq \underline{a}_j$.

12.2 We note that Λ is uniquely determined up to its dimensionality n, by giving Σ. Furthermore condition (12:2b) is equivalent to the following:

(12:2b1) Let $\underline{a}$ be any n-dimensional vector of non-negative integral components satisfying $\underline{a}_i \geq \alpha$ where $[\alpha, i]$ is in Σ. Let $[\beta, j]$ be in Σ where $[\beta, j] \leq [\alpha, i]$. Then $\underline{a}_j \leq \beta - 1$ implies $\underline{a}$ does not exist in Λ.

or we may alternately restate (12:2b1) as:

(12:2b2) If $[\beta, j] \leq [\alpha, i]$ then for every vector $\underline{a}$ in Λ satisfying $\underline{a}_j \leq \beta - 1$ we must have $\underline{a}_i \leq \alpha - 1$.

We note that (12:2b2) is precisely the definition of $[\beta, j]$ causes $[\alpha, i]$, (11:8). Thus we obtain our first insight into the relationship between Λ and lattices generated by circuits. The first result to be established is that Λ is a lattice under the numerical operations $\vee$, $\wedge$ and $\leq$((6:4), (6:5), (6:6)).

(12:A) Λ is a distributive lattice with a $\underline{0}$ under the operations $\vee$, $\wedge$, and $\leq$.

<u>Proof:</u> Clearly the vector $\underline{0} = [0, 0, \ldots, 0]$ is in Λ since it has no component $\underline{a}_i > 0$ and thus both (12:2a) and (12:2b) are trivially satisfied. Moreover $\leq$ is a partial ordering in Λ, hence $\underline{0}$ is clearly a lower bound for all vectors $\underline{a}$ in Λ with respect to this partial ordering.

Now let $\underline{a}$ and $\underline{b}$ be two vectors in Λ. Since $\underline{a} \wedge \underline{b}$ and $\underline{a} \vee \underline{b}$ have components chosen from either $\underline{a}$ or $\underline{b}$ then (12:2a) holds for both of them. Let $\underline{c} = \underline{a} \vee \underline{b}$ and assume $\underline{c}_i = \underline{a}_i > 0$. Let $\underline{c}_j$ denote any other component of $\underline{a} \vee \underline{b}$. Then if $\underline{c}_j = \underline{a}_j$ condition (12:2b) is satisfied as a consequence of it holding

for the vector $\underline{a}$. On the other hand if $\underline{c}_j = \underline{b}_j$ then by the properties of $\vee$, $\underline{a}_j \leq \underline{b}_j$. Hence (12:2b) also holds as a consequence of it holding for the vector $\underline{a}$ and Λ is thus closed under the operation $\vee$.

Now let $\underline{c} = \underline{a} \wedge \underline{b}$ and assume $\underline{c}_i = \underline{a}_i > 0$. By the properties of $\wedge$ we have $\underline{a}_i \leq \underline{b}_i$ thus $\underline{b}_i > 0$ and by (12:2a) and (12:1b) we further obtain $[\underline{a}_i, i] \leq [\underline{b}_i, i]$. Let $\underline{c}_j$ be any other component of $\underline{a} \wedge \underline{b}$. If $\underline{c}_j = \underline{a}_j$ then (12.2b) is satisfied as a consequence of it holding for the vector $\underline{a}$. On the other hand let $\underline{c}_j = \underline{b}_j$ and assume that some $[\gamma, j]$ exists satisfying both $[\gamma, j] \leq [\underline{c}_i, i] = [\underline{a}_i, i]$ and $\gamma > \underline{c}_j = \underline{b}_j$. Thus $\underline{c}$ cannot be in Λ by (12:2b). But since $[\underline{a}_i, i] \leq [\underline{b}_i, i]$ we have that $[\gamma, j]$ is also $\leq [\underline{b}_i, i]$ hence (12:2b) does not hold for the vector $\underline{b}$. Thus we conclude that $\underline{c} = \underline{a} \wedge \underline{b}$ is in Λ and Λ is therefore closed under the operation $\wedge$. However the operations $\vee$ and $\wedge$ are respectively the least upper and greatest lower bounds with respect to the partial order $\leq$. Therefore Λ is a lattice. Furthermore it must also be a distributive lattice since these operations are distributive over vectors with integral components.

The next result is needed in order to apply Birkhoff's theorem to the lattice Λ (Ref. 3 p. 142-3; Section 10 p. 49).

(12:3) For any vector $\underline{a}$ in Λ there is only a finite number of vectors $\underline{b}$ in Λ satisfying $\underline{b} < \underline{a}$.

<u>Proof:</u> Since the components of $\underline{a}$ are all finite and ≥ 0, the number of all possible vectors, $\underline{b}$, with non-negative integral components satisfying $\underline{b} < \underline{a}$ is finite. Hence any subset of them, in particular those $\underline{b}$ in Λ such that $\underline{b} < \underline{a}$, is also finite.

Next we establish that the covering relationship in Λ is precisely the same relationship that holds in lattices arising from semi-modular circuits.

(12:4) $\underline{a}$ is covered by $\underline{b}$ in Λ if and only if $\underline{b}_j = \underline{a}_j + 1$ for some j and $\underline{a}_i = \underline{b}_i$ for all $i \neq j$.

<u>Proof:</u> If $\underline{b}_j = \underline{a}_j + 1$ and $\underline{b}_i = \underline{a}_i$ for $i \neq j$ then $\underline{a} < \underline{b}$. Further since the elements of Λ can only possess integral components, there does not exist an element $\underline{c}$ satisfying $\underline{a} < \underline{c} < \underline{b}$, hence $\underline{b}$ covers $\underline{a}$.

Let $\underline{b}$ cover $\underline{a}$ in Λ; thus $\underline{b} > \underline{a}$ and we may assume that for a subset of the components $i(j)$, $j = 1, 2, \ldots, k$ we have $\underline{b}_{i(j)} > \underline{a}_{i(j)}$. For the remaining components $i \neq i(j)$, we have $\underline{b}_i = \underline{a}_i$. Hence by (12:1b) the elements $[\underline{a}_{i(1)} + 1, i(1)]$, $[\underline{a}_{i(2)} + 1, i(2)], \ldots, [\underline{a}_{i(k)} + 1, i(k)]$ are all in Σ. Since Σ is partially ordered, for some m, $1 \leq m \leq k$, the element $[\underline{a}_{i(m)} + 1, i(m)]$ has the property that for any $j \neq m$ and $1 \leq j \leq k$, $[\underline{a}_{i(j)} + 1, i(j)] \not< [\underline{a}_{i(m)} + 1, i(m)]$. The element $[\underline{a}_{i(m)} + 1, i(m)]$ is not, of course, necessarily unique. Consider the vector $\underline{c}$ defined as $\underline{c}_{i(m)} = \underline{a}_{i(m)} + 1$ and $\underline{c}_j = \underline{a}_j$ for $j \neq i(m)$. If $\underline{c}$ were in Λ then we would have the desired result for then, $\underline{a} < \underline{c} \leq \underline{b}$ and $\underline{b}$ covering $\underline{a}$ implies $\underline{b} = \underline{c}$.

If now, $\underline{c}$ is not in Λ then there exist $[\beta, i]$ and $[\underline{c}_p, p]$ in Σ such that $[\beta, i] \leq [\underline{c}_p, p]$ and $\beta > \underline{c}_i$. We will show firstly that $p = i(m)$. For assume p were not $i(m)$, then $\underline{c}_p = \underline{a}_p$. But $\underline{c}_j \geq \underline{a}_j$ for all j hence $\beta > \underline{a}_i$ and by (12:2b) $\underline{a}$ could not be in Λ. Next we show that i has to be one of the $i(j)$. Assume it were not then we would have $\underline{a}_i = \underline{c}_i = \underline{b}_i$. But $[\underline{a}_{i(m)} + 1, i(m)] \leq [\underline{b}_{i(m)}, i(m)]$ hence $[\beta, i] \leq [\underline{b}_{i(m)}, i(m)]$ and $\beta > \underline{c}_i = \underline{b}_i$. Thus by (12:2b) the vector $\underline{b}$ could not be in Λ, hence i must be one of the $i(j)$. Moreover i cannot be $i(m)$ since if it were, then the fact that $p = i(m)$ would imply by (12:2a) that $\beta \leq \underline{c}_{i(m)}$. But i being $i(m)$ would also imply $\beta > \underline{c}_{i(m)}$ which is a contradiction. Thus i is some component $i(j)$ where $j \neq m$ and $1 \leq j \leq k$. Further since $\beta > \underline{c}_{i(j)} = \underline{a}_{i(j)} = \underline{c}_i$ we have by application of (12:1b) that for some component $i(j)$, $j \neq m$, $[\underline{a}_{i(j)} + 1, i(j)] \leq [\beta, i(j)] \leq [\underline{a}_{i(m)} + 1, i(m)] = [\underline{c}_p, p]$. But this contradicts the original choice of the component $i(m)$, hence $\underline{c}$ must lie in Λ which completes the proof.

12.3 So far we have succeeded in producing a Λ from a given Σ. We wish now to show that a given Λ has a unique image Σ. That is if Σ_1 and Σ_2 both give rise to the same lattice Λ then $\Sigma_1 = \Sigma_2$. Before we may establish this we must establish two lemmas. The first of these states the correspondence between Σ and the join-irreducible elements of the lattice Λ.

(12:5) The set of non-zero join-irreducible elements may be mapped one to one into the change chart Σ. Under this mapping $\underline{a}$ corresponds to $[\alpha, i]$ if and only if $\underline{a}$ is the minimal element in Λ satisfying $\underline{a}_i = \alpha$.

(12:5a) Furthermore if $\underline{a}_j > 0$ for any j, then $[\underline{a}_j, j] \leq [\alpha, i]$ and $[\underline{a}_j + 1, j] \not\leq [\alpha, i]$ whenever $[\underline{a}_j + 1, j]$ exists in Σ.

<u>Proof:</u> Let $\underline{a}$ be any non-zero join-irreducible element. Since $\underline{a} \neq 0$ there exists at least one element $\underline{b}$ such that $\underline{a}$ covers $\underline{b}$. Furthermore if there existed another such element $\underline{c}$, then $\underline{a} = \underline{b} \vee \underline{c}$ and hence $\underline{a}$ would not be join-irreducible. Therefore $\underline{a}$ covers only a single element $\underline{b}$ which differs from $\underline{a}$ in a unique component i. In addition since $\underline{a}_i = \underline{b}_i + 1$ then $\underline{a}_i \neq 0$ hence $[\underline{a}_i, i]$ exists in Σ. Now let $\underline{d}$ be any other element such that $\underline{d}_i = \underline{a}_i$ and form the vector $\underline{a} \wedge \underline{d}$. Since $\underline{a} \wedge \underline{d} \leq \underline{a}$ we may construct a covering sequence $\underline{a} \wedge \underline{d}$, $\underline{d}(1)$, $\underline{d}(2), \ldots, \underline{d}(k)$, $\underline{a}$. By the above argument $\underline{d}(k) = \underline{b}$ hence $\underline{d}(k)_i = \underline{a}_i - 1$. But $(\underline{a} \wedge \underline{d})_i = \underline{a}_i$ hence to avoid a contradiction we must have $\underline{a} \wedge \underline{d} = \underline{a}$, or $\underline{a} \leq \underline{d}$. Thus $\underline{a}$ is the minimal vector in Λ whose $i^{\underline{th}}$ component is $\underline{a}_i$.

Now let $\underline{a}$ be a vector which is minimal with respect to some positive component $\underline{a}_i$. This vector must exist for if $\underline{b}$ and $\underline{c}$ were both minimal with respect to $\underline{b}_i = \underline{c}_i = \underline{a}_i$ then $\underline{b} \wedge \underline{c}$ would be minimal with respect to both $\underline{b}$ and $\underline{c}$. Further $\underline{a} \neq 0$ since $\underline{a}_i \neq 0$, and in addition it can only cover a vector $\underline{b}$ which differs from $\underline{a}$ in the ith component, otherwise $\underline{a}$ would not be minimal with respect to this component. Thus $\underline{a}$ must be join-irreducible since it cannot cover two or more distinct states.

At this point we have shown that every non-zero join-irreducible element corresponds to some element in Σ. We have not shown that this mapping exhausts all of Σ, that is there might be elements of Σ for which no join-irreducible element in Λ is an image. Thus the mapping is into. Finally the mapping is 1-1 for let both $\underline{a}$ and $\underline{b}$ correspond to $[\alpha, i]$. Then $\underline{a}_i = \underline{b}_i = \alpha$ and hence $(\underline{a} \wedge \underline{b})_i = \alpha$. But $\underline{a}$ is minimal with respect to $\underline{a}_i = \alpha$ hence $\underline{a} \wedge \underline{b} = \underline{a}$. Similarly $\underline{b}$ is also minimal hence $\underline{a} \wedge \underline{b} = \underline{b} = \underline{a}$.

To establish (12:5a) we let $\underline{a}_{j(1)}, \underline{a}_{j(2)}, \ldots, \underline{a}_{j(k)}$ be all the other non-zero components of $\underline{a}$. Denying (12:5a) we obtain by finite induction, there exists an m such that $[\underline{a}_{j(m)}, j(m)] \not\leq [\underline{a}_i, i]$ and $[\underline{a}_{j(m)}, j(m)] \not\leq [\underline{a}_{j(s)}, j(s)]$ for all $s \neq m$. Construct the vector $\underline{b} = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_{j(m)} - 1, \ldots, \underline{a}_n)$. We will show that $\underline{b}$ lies in Λ thus violating the condition that $\underline{a}$ is the minimal vector in Λ whose i^{th} component is $\underline{a}_i$.

Condition (12:2a) holds trivially. To establish (12:2b) we note firstly that since $\underline{a}$ is in Λ then for any non-zero component $\underline{a}_t$ and any $[\beta, j]$ satisfying $[\beta, j] \leq [\underline{a}_t, t]$ we have $\beta \leq \underline{a}_j$. In particular if $j = j(m)$, then since $[\underline{a}_{j(m)}, j(m)] \not\leq [\underline{a}_t, t]$ for all non-zero components $\underline{a}_t$, we would have $\beta \leq \underline{a}_{j(m)} - 1 = \underline{b}_{j(m)}$. Thus since $\underline{a}_t = \underline{b}_t$ for $t \neq j(m)$ we have condition (12:2b) holding for all the non-zero components of $\underline{b}$ except $j(m)$. If $\underline{b}_{j(m)} = 0$ then we have the desired result hence we need to only establish (12:2b) for the case $\underline{b}_{j(m)} \neq 0$. However in this case since $[\underline{b}_{j(m)}, j(m)] = [\underline{a}_{j(m)} - 1, j(m)] < [\underline{a}_{j(m)}, j(m)]$, then any $[\beta, j] \leq [\underline{b}_{j(m)}, j(m)]$ would satisfy $[\beta, j] \leq [\underline{a}_{j(m)}, j(m)]$. Hence (12:2b) holds for the $j(m)$ component as a result of it holding for $\underline{a}_{j(m)}$. Therefore the vector $\underline{b}$ lies in Λ which was the desired objective. Thus the first condition of (12:5a) holds.

Finally we note that if $[\underline{a}_j + 1, j]$ existed in Σ and satisfied $[\underline{a}_j + 1, j] \leq [\underline{a}_i, i]$, the vector $\underline{a}$ would violate (12:2b) hence (12:5a) is established which completes the proof.

The next lemma states that the mapping of (12:5) is an "onto" mapping. That is for every $[\alpha,i]$ in Σ there corresponds a join-irreducible element $\underline{a}$ in Λ which is minimal with respect to $\underline{a}_i = \alpha$. In light of this fact, (12:5a) is quite important for it allows us to easily construct directly from the change chart every non-zero join-irreducible element of Λ. The lattice Λ is then obtained by forming all possible joins of join-irreducible elements. This is a far easier method of constructing Λ than any method which forms each individual element by verifying conditions (12:2a) and (12:2b).

(12:6) The set of non-zero join-irreducible elements is isomorphic to Σ where $[\alpha,i] \leq [\beta,j]$ if and only if $\underline{a} \leq \underline{b}$.

<u>Proof:</u> To prove the mapping is "onto" we need only show that for any $[\alpha,i]$ of Σ a vector $\underline{a}$ exists in Λ with $\underline{a}_i = \alpha$. From this result we obtain the join-irreducible element by taking the minimum vector in Λ with i^{th} component α. We begin by letting $[\beta,j]$ be any element of Σ such that $[\beta,j] \leq [\alpha,i]$ for $j \neq i$. If no such $[\beta,j]$ exists then the vector $\underline{a} = (0,0,\ldots,\underline{a}_i,\ldots,0)$ where $\underline{a}_i = \alpha$ satisfies both (12:2a) and (12:2b). If some such $[\beta,j]$ does exist then since Σ satisfies (12:1a) there exists a $[\gamma,j]$ in Σ such that $[\gamma,j] \leq [\alpha,i]$ and $[\gamma+1,j]$ (if it exists) $\not\leq [\alpha,i]$. Let the set of all of these $[\gamma,j]$ be denoted by $[\gamma_k,j(k)]$. Consider the vector $\underline{a}$ defined by $\underline{a}_i = \alpha$ and $\underline{a}_{j(k)} = \gamma_k$ for all components $j(k)$. For any other component $r \neq i$ or $\neq j(k)$ we take $\underline{a}_r = 0$. If $\underline{a}$ is not in Λ then since condition (12:2b) holds for $[\underline{a}_i,i]$ it must fail on one of the $[\gamma_k,j(k)]$. That is there exists a $[\beta,j] \leq [\gamma_m,j(m)] = [\underline{a}_{j(m)},j(m)]$ such that $\beta > \underline{a}_j$. But by (12:1b) we have $[\underline{a}_j + 1, j] \leq [\beta,j]$ and by the choice of $[\gamma_k, j(k)]$ we have $[\gamma_m, j(m)] \leq [\underline{a}_i,i]$, hence $[\underline{a}_j + 1, j] \leq [\underline{a}_i,i]$. But $[\underline{a}_j,j]$ was one of the $[\gamma_k, j(k)]$ and was chosen so that $[\underline{a}_j + 1, j] \not\leq [\underline{a}_i,i]$ when it existed. Thus we have a contradiction hence $\underline{a}$ is in Λ. Furthermore since $\underline{a}_i = \alpha$ the set of all

vectors having $i^{\underline{th}}$ component α is non-empty, we then may take the minimum vector of this set which is join-irreducible by (12:5). (Moreover by (12:5a) this join-irreducible element is precisely the vector $\underline{a}$ constructed in the above argument). Thus the mapping is both 1-to-1 and onto and we need now only show the isomorphism between the set of non-zero join-irreducible elements and Σ.

Assume $[\alpha,i] \leq [\beta,j]$ and let $\underline{a}$ and $\underline{b}$ be the join-irreducible elements corresponding to them. By (12:2b) we have $\alpha \leq \underline{b}_i$ hence $\underline{a} \wedge \underline{b}$ has an ith component equal to α. However $\underline{a}$ is the minimum of such states thus $\underline{a} \wedge \underline{b} = \underline{a}$ or hence $\underline{a} \leq \underline{b}$. Now let $\underline{a} \leq \underline{b}$. Since $\alpha = \underline{a}_i \leq \underline{b}_i > 0$ we may apply (12:5a) to the vector $\underline{b}$ obtaining $[\underline{b}_i,i] \leq [\beta,j]$. But $[\underline{a}_i,i] \leq [\underline{b}_i,i]$ by (12:1b), hence $[\alpha,i] \leq [\beta,j]$.

We now may establish the uniqueness of the image of a given Λ in the set of all change charts.

(12:B) If Σ_1 and Σ_2 both give rise to the same lattice Λ then $\Sigma_1 = \Sigma_2$.

<u>Proof:</u> (12:6) has established that both Σ_1 and Σ_2 are isomorphic to each other. Let $[\alpha,i]$ in Σ_1 correspond to $[\beta,j]$ in Σ_2 where both of them map onto the same join-irreducible element $\underline{a}$. By (12:5) we must have $i = j$ since $\underline{a}$ cannot be minimal with respect to two components. Furthermore $i = j$ implies $\alpha = \beta = \underline{a}_i$ which completes the proof.

12.4 Previously in chapter II we introduced the causation signals for a distributive lattice D:C[u]. We remark, however, that it is not necessary for the lattice to be generated by a set of "circuit" functions starting in a given initial state. Indeed only a few conditions need be satisfied in order to define these causation signals, and for the sake of mathematical elegance, we will use these in this section. We begin by defining a set Λ which is to take the place of D:C[u].

(12:7) Λ is a set of finite dimensional vectors with non-negative integral components containing the vector $\underline{0} = (0,0,\ldots,0]$ and satisfying:

(12:7a) With respect to the partial ordering $\leq$, the vector $\underline{b}$ covers $\underline{a}$ in Λ if and only if $\underline{b}_j = \underline{a}_j + 1$ for some j and $\underline{b}_i = \underline{a}_i$ for $i \neq j$.

(12:7b) Λ is closed with respect to the operations $\wedge$ and $\vee$.

We see immediately that Λ satisfies theorems (12:A) and (12:3). Furthermore we may take over the definition of a causation signal (11:7), their partial ordering (11:8) and (11:B), and finally theorem (11:C), which states that the set of causation signals (on Λ and not just the more restricted lattice of a distributive circuit) satisfies (12:1). We are unable to take over any cycling theory since it arose from a study of the I-states and hence is functionally dependent upon the circuit and not its lattice.

From the above remarks we see that we have been able to map a given Λ satisfying (12:7) onto a partially ordered set satisfying (12:1). Earlier we mapped a given Σ satisfying (12:1) onto a set satisfying (12:7). Furthermore we have shown by (12:B) that if Σ_1 and Σ_2 could be mapped onto the same set Λ then $\Sigma_1 = \Sigma_2$. The inverse of (12:B) does not hold since clearly a given Σ could be mapped onto many different Λ_i depending on the dimension. Its image is thus only unique when the dimension of Λ is fixed. In order to show that sets satisfying (12:7) and (12:1) are completely equivalent, we need only prove one more theorem.

(12:C) Let Λ be the image of a set Σ defined by (12:2). Let Σ_1 be the image of Λ defined by (11:7) and (11:8). Then $\Sigma_1 = \Sigma$.

<u>Proof:</u> By (12:6) and (11:B) Σ_1 is isomorphic to Σ since they are both ismorphic to the same set of join-irreducible elements. Let $[\alpha, i]$ and $[\alpha_1, j]$ be isomorphic elements of Σ and Σ_1 and let the join-irreducible element corresponding to both of them be $\underline{a}$. Since $\underline{a}$ is both minimal with respect to $\underline{a}_i = \alpha$ by (12:6) and with respect to $\underline{a}_j = \alpha_1$ by (11:B) it follows that we must have $i = j$ and $\alpha = \alpha_1$ to avoid a contradiction.

12.5 At this point we have completed the first step of the synthesis procedure by showing that sets satisfying (12:7) and sets satisfying (12:1) are completely equivalent. True, sets satisfying (12:7) are a larger class of sets than distributive circuits but at the same time sets satisfying (12:1) are also a larger class than sets of causation signals of distributive circuits. This follows from the fact that a set of causation signals $\Sigma[u]$ of a distributive circuit D:C[u] must further satisfy theorem (11:E). Thus, if a given change chart Σ is mapped onto a lattice λ for which there existed a distributive circuit D:C[u] equal to λ, then Σ would be equal to some $\Sigma[u]$ and the conditions of theorem (11:E) would apply to Σ. The task now before us is: given a change chart Σ, prove the sufficiency of the conditions in theorem (11:E). That is: find a set of functions which when started in an initial state u, yields a distributive lattice D:C[u] equal to λ, the image of the given change chart Σ. Although it will not be possible to establish the complete sufficiency of (11:E), we will be able to prove a slightly weaker result. We will show the sufficiency of (11:E) provided we may add new nodes to the given change chart. These additions will not upset any of the partial orderings in the original change chart. Hence the behavior of the final circuit on the subset of the nodes excluding the new nodes, is precisely the behavior desired.

Since we are now speaking of more general lattices we must firstly define what we mean by excitation for the nodes or components of vectors in these lattices. Furthermore the definition must be compatible with the previous concept of excitation. In addition we must investigate the structure of the change chart when a particular node is excited or in equilibrium. The first result to be established towards this end is a general lattice theoretic one.

(12:8) Let x be join-irreducible in a distributive lattice L which satisfies the condition that all decending chains are finite. Let $y_1, y_2, \ldots, y_k$ be the set of join-irreducible elements in L which are covered by x in the partially ordered set, J, of join-irreducible elements of L. Then z is covered by x in L if and only if z has as its irredundant join of join-irreducible elements the representation $z = y_1 \cup y_2 \cup \cdots \cup y_k$.

<u>Proof:</u> Let z be covered by x in L and let its irredundant representation be $y_1 \cup y_2 \cup \ldots y_k$ (See Birkoff p. 143). Since $z < x$ then $y_i < x$ for each $i = 1, 2, \ldots, k$. Let w be in J and $w \leq x$, then $z \leq z \cup w \leq x$. But z is covered by x hence we have either $z \cup w = x$ or $z \cup w = z$. If $z \cup w = x$ then since x is join-irreducible we have x equal to one of the y_i or equal to w. However $x \neq y_i$ hence $w = x$. If on the other hand $z \cup w = z$ then we have $w \leq y_1 \cup y_2 \cup \ldots \cup y_k$. But since w is join-irreducible, $w \leq y_i$ for all i (Lemma 1, p. 139, Ref. 3).

We prove now that x covers all of the y_j in J. Assume that $y_j \leq w$ in the above paragraph. In this case we have either $w = x$ or $w \leq y_i$ for some i. However the representation of z is irredundant;hence if $w \leq y_i$, we must have $y_j = y_i$ or $w = y_j$. Therefore we now need only show that all the elements covered by x in J are contained in the y_i. Assume therefore that w is covered by x in the above paragraph. Since $w \neq x$ we must have $w \leq y_i$ for some i. However w is covered by x in J and $y_i < x$ hence we have $w = y_i$ for some i, which completes the proof.

(12:9) Let $\underline{a}$ be the join-irreducible element corresponding to $[\alpha, i]$ of a change chart Σ. Then $\underline{a}_j = \beta$ if and only if $[\beta, j] \leq [\alpha, i]$ in Σ whenever $\beta > 0$ and $[\beta + 1, j] \not\leq [\alpha, i]$ when $[\beta + 1, j]$ exists in Σ.

Proof: (12:5a) establishes the theorem in one direction hence we need only prove the other. Let $[\beta, j] \leq [\alpha, i]$ for $\beta > 0$ then the vector $\underline{b} \sim [\beta, j]$ satisfies $\underline{b} \leq \underline{a}$ by (12:6). Hence $\underline{a}_j \geq \underline{b}_j = \beta$. Furthermore if $[\beta + 1, j]$ does not exist then for any vector $\underline{c}$ in Λ, $\underline{c}_j \leq \beta$. Therefore we have $\underline{a}_j = \beta$. If on the other hand $[\beta + 1, j]$ exists and is $\not\leq [\alpha, i]$, and if $\underline{c} \sim [\beta + 1, j]$, then by (10:8) either $\underline{c} \leq \underline{a}$ or $\underline{a}_j \leq \beta$. If $\underline{c} \leq \underline{a}$ then we have $[\beta + 1, j] \leq [\alpha, i]$ which is a contradiction. Thus $\underline{a}_j \leq \beta$; hence by the previous calculation we have $\underline{a}_j = \beta$.

12.6 We note that in the proof of (12:9) we used a result developed for C-signals of a distributive circuit. The remarks of paragraph 11.4 allow us to translate this to a result about causation signals of a distributive circuit however it is not clear why this should apply to the more general set of lattices λ. This becomes clear only when we examine the proof of (10:8) and discover that it depended only upon the properties of the lattice of C-states and not upon the properties of the I-states. Next we define excitation so as to remove it from any functional dependence upon a circuit. We will use, from now on, node and component interchangably.

(12:10) Node i is said to be excited in a vector $\underline{a}$ in Λ if and only if there exists a vector $\underline{b}$ in Λ covering $\underline{a}$ such that $\underline{b}_i = \underline{a}_i + 1$.

This definition is more general than the usual definition as it applies to the more general Λ and is consistent with the previous definition when Λ is restricted to a distributive circuit. The next theorem is quite important to the synthesis procedure as it provides a method of predicting from the change chart whether a given node of a particular vector is excited.

(12:D) Let $\underline{a}$ be in Λ and Σ its change chart. Then the ith node is excited in $\underline{a}$ if and only if $[\alpha + 1, i]$ exists and $\underline{a}_{j(k)} \geq \beta_k$ whenever $[\beta_k, j(k)]$ is covered by $[\alpha + 1, i]$.

Proof: Let $\underline{b}$ be the unique element covered by the join-irreducible element $\underline{c} \sim [\alpha + 1, i]$. Then by (12:5), (12:8) and (12:9) we obtain $\underline{b}_{j(k)} = \beta_k$, $\underline{b}_i = \alpha$ and $\underline{b} = \underline{b}(1) \cup \underline{b}(2) \cup \ldots \cup \underline{b}(k)$ where $\underline{b}(r) \sim [\beta_r, j(r)]$. If we assume $\underline{a}_{j(k)} \geq \beta_k$ for all β_k then since $\underline{b}(r) \sim [\beta_r, j(r)]$ for all r, we have $\underline{b}(r) \leq \underline{a}$ by (12:5) and hence $\underline{b} \leq \underline{a}$. Now consider the state $\underline{a} \vee \underline{c}$. Since $\underline{a}_i = \alpha$, then $\underline{c} \not\leq \underline{a}$, further $\underline{c}$ covers $\underline{b}$ hence $\underline{a} \vee \underline{c}$ covers $\underline{a}$ (See Ref. 3, p. 100). However $\underline{a} \vee \underline{c}$ has $i^{\underline{th}}$ component $\alpha + 1$ and $\underline{a}$ has $i^{\underline{th}}$ component α hence node i is excited in $\underline{a}$.

If now the ith node is excited in state $\underline{a}$ then there exists a state $\underline{d}$ covering $\underline{a}$ where $\underline{d}_i = \alpha + 1$ and $\underline{d}_j = \underline{a}_j$ for $j \neq i$. Letting $\underline{c} \sim [\alpha + 1, i]$ we have $\underline{d} \geq c$. Hence by (12:9) $\underline{d}_{j(k)} = \underline{a}_{j(k)} \geq \beta_k$ where $[\beta_k, j(k)]$ is any signal covered by $[\alpha + 1, i]$.

12.7 We now turn our attention to finding a set of functions which, when started in an initial state, generates a distributive lattice possessing a set of causation signals equal to a given change chart. We begin by defining a state assignment of a lattice Λ. This assignment is nothing more than a restricted mapping carrying Λ onto a set S of finite dimensional vectors whose components take on a finite number of positive integral values. We will show that this assignment is uniquely characterized by only the state value a_i of the join-irreducible element $\underline{a}$ corresponding to $[\alpha, i]$ for each $[\alpha, i]$ in Σ.

(12:11) Given an initial state u, a set S is said to be a state assignment of a lattice Λ if and only if S is a set of n-dimensional vectors a satisfying:

(12:11a) for each component i there exists an integer k_i such that $0 \leq a_i < k_i$ for all a in S and,

(12:11b) there is a mapping carrying every $\underline{a} \neq \underline{0}$ in Λ onto a unique state a in S, and carrying $\underline{0}$ in Λ onto u in S, such that: if $\underline{0}$, $\underline{a}(1)$, $\underline{a}(2)$, ..., $\underline{a}$ is a covering sequence in Λ and u, a(1), a(2),..., a its image in S, then $\underline{a} = L\,[u, a(1), a(2), \ldots, a]$.

We note that in (12:11b) we used the concept of a length vector. Although it was originally defined on a partial allowed sequence, it can be easily extended to include any sequence of n-dimensional vectors with non-negative integral components. Also we have restricted the vectors in S to possessing the same dimension as those in Λ. Although this requirement is not absolutely necessary (the only restriction is that the dimension of the vectors in S must be at least the dimension of the vectors in Λ), no loss of generality occurs in making it. From (12:11b) we infer the following property:

(12:12) If $\underline{a} \leq \underline{b}$ in Λ and $\underline{a}_i = \underline{b}_i$ then $a_i = b_i$ where a and b are the assignments of $\underline{a}$ and $\underline{b}$ respectively.

<u>Proof:</u> Take the covering sequence $\underline{0}$, $\underline{a}(1)$, $a(2),\ldots,$ $\underline{a}$, $\underline{b}(1)$, $\underline{b}(2),\ldots,$ $\underline{b}$ then if u, $a(1)$, $a(2),\ldots,$ a, $b(1)$, $b(2),\ldots,$ b is its image in S we have $(L\,[a, b(1), b(2),\ldots, b])_i = 0$ hence $a_i = b_i$.

Next we show that we need only specify the values a_i of every $\underline{a} \sim [\alpha, i]$ in the change chart.

(12:13) Let u be the initial state and S an assignment of the lattice Λ. Let $\underline{a}$ be any vector in Λ, then the assigment of $\underline{a}$ is given by:

$$a_i = u_i \text{ if } \underline{a}_i = 0$$

$$a_i = b_i \text{ where b is the assignment of } \underline{b} \sim [\underline{a}_i, i].$$

<u>Proof:</u> Let $\underline{a}_{i(j)}$, $j = 1, 2,\ldots, k$ be all the non-zero components of $\underline{a}$. Let $\underline{b}(j)$ be the join-irreducible elements corresponding to $[\underline{a}_{i(j)}, i(j)]$ respectively. Then since $\underline{b}(j)$ is minimal with respect to vectors possessing $i(j)^{\underline{th}}$ component equal to $\underline{a}_{i(j)}$ we have $\underline{b}(j) \leq \underline{a}$ for all j. Hence by (12:12) $a_{i(j)} = b(j)_{i(j)}$ where a is the assignment of $\underline{a}$ and $b(j)$ the assignment of $\underline{b}(j)$. For those components of $\underline{a}$ satisfying $\underline{a}_i = 0$ we again use (12:12) to obtain $a_i = u_i$ since $\underline{0} \leq \underline{a}$.

We note that (12:12) along with the definition of the length vector implies: if $\underline{a} \sim [\alpha, i]$ and $\underline{b} \sim [\alpha + 1, i]$ then either $b_i = a_i + 1$ or $b_i = a_i - 1$. Hence from (12:13) and this last remark, any assignment may be accomplished directly from the change chart and the initial state. We thus need not assign a state a in S for every state $\underline{a}$ in Λ, but only a state value a_i for every $[\alpha, i]$ in Σ. This assignment must, of course, be consistent with the requirements that a_i be chosen from an interval $0 \leq a_i < k_i$ and that a_i either equals $b_i - 1$ or $b_i + 1$ where b_i was the assignment of $[\alpha - 1, i]$ when $\alpha - 1 > 0$. In the case where $\alpha - 1 = 0$ then we require a_i to be either $u_i - 1$ or $u_i + 1$.

12.8 We now are in a position to define a set of functions from an assignment S, the initial state u and the lattice Λ. We recall that (12:D) gave us the excitation conditions for nodes in the lattice Λ. Using this condition we infer immediately that if the ith node is excited in the state $\underline{a}$ then there exists a state $\underline{b}$ covering $\underline{a}$ such that $\underline{b}_i = \underline{a}_i + 1$. In this case we will have that the function z_i' describing the ith node must satisfy either $z_i' \geq a_i + 1$ or $z_i' \leq a_i - 1$ depending on whether $b_i = a_i + 1$ or $b_i = a_i - 1$ respectively. In particular we may, if we wish, choose $z_i' = b_i$. Thus for each state $\underline{a}$ in Λ we may determine firstly by (12:D) whether a particular node i is excited or not. Then if this node is in equilibrium, the value of z_i' is given by $z_i' = a_i$. If, however, the ith node is excited and $\underline{b}$ is that state covering $\underline{a}$ where $\underline{b}_i = \underline{a}_i + 1$ then we choose $z_i' = b_i$. This particular type of function assignment will be called the covering function assignment. We note that this assignment is carried out on every vector $\underline{a}$ in Λ hence in the case where Λ is infinite the functions z_i' are defined on an infinite set of vectors. However, since the number of states in a state assignment is finite we will have, in general, a number of different states in Λ mapped onto the same state in S. Thus if say $\underline{a}$ and $\underline{b}$ are mapped onto the same state a in an assignment S then we hope that z_i' when computed for

the state $\underline{a}$ and z_i' when computed for the state $\underline{b}$ would be equal for all $i = 1, 2, \ldots, n$. If they are not equal then since the covering function assignment is determined from the state assignment, we will say that the state assignment was inconsistent. Before we rigorously define the covering function assignment we note the following property. Let $\underline{a}$ be any vector in Λ and let $\underline{b}(1), \underline{b}(2), \ldots, \underline{b}(k)$ be all the vectors in Λ covering $\underline{a}$. Then the vector $\underline{b}(\underline{a}) = \underline{b}(1) \vee \underline{b}(2) \vee \ldots \vee \underline{b}(k)$ has the property that $\underline{b}_i = \underline{a}_i + 1$ if and only if the $i^{\underline{th}}$ node was excited in the vector $\underline{a}$. Otherwise $\underline{b}_i = \underline{a}_i$ if the ith node was in equilibrium in the vector $\underline{a}$. Moreover by (12:12) if $\underline{b}_i = \underline{a}_i + 1$ then $b_i = b(j)_i$ where $\underline{b}(j)$ is that state covering $\underline{a}$ whose i^{th} component is $\underline{a}_i + 1$. Thus if $z'(\underline{a})$ is the vector denoting the values of the covering function assignments for the vector $\underline{a}$ then $z'(\underline{a}) = b(\underline{a})$ where $b(\underline{a})$ is the state assignment of $\underline{b}(\underline{a})$. This property will be used in the definition of the covering function assignment.

(12:14) For each vector $\underline{a}$ in Λ we define the vector $z'(\underline{a})$ for the state assignments S as $z'(\underline{a}) = b(\underline{a})$ where $b(\underline{a})$ is the state assignment of the vector $\underline{b}(\underline{a}) = \underline{b}(1) \vee \underline{b}(2) \vee \ldots \vee \underline{b}(k)$ and $\underline{b}(j)$, $j = 1, 2, \ldots, k$ is the set of all vectors in Λ covering $\underline{a}$. The covering function assignment then is the set of all such vectors $z'(\underline{a})$ where $\underline{a}$ is in Λ.

12.9 Next we define the concept of consistency. Here we will use the notation S[u] to mean the set of all states in a state assignment.

(12:15) The state assignment S[u] is said to be consistent if and only if $Z'(\underline{a}) = Z'(\underline{b})$ for every pair of vectors $\underline{a}$ and $\underline{b}$ in Λ such that $\underline{a}$ and $\underline{b}$ both map onto the same state a in S[u].

We note that if a particular state assignment S[u] is consistent, the covering function assignment may now be defined only for the states in S[u]. That is, z' is now only a function of the states in S[u] and hence only a function of a finite number of states. Furthermore we note that if a is any state in S[u],

it arose from some vector $\underline{a}$ in Λ, hence z' is defined for that state. In addition since the vector $\underline{b}(a)$ in (12:14) is in Λ and since $z' = b(a)$ in S[u], we have that z' is a mapping of S[u] onto itself. Thus given Λ and a consistent state assignment S[u] of Λ, we can determine a set of n-functions $z_i' = f_i(z_1, z_2, \ldots, z_n)$ mapping S[u] onto itself. What we wish to show is that these functions when considered as a circuit will generate Λ starting in the initial state u.

(12:16) Let S[u] be a consistent state assignment of a lattice Λ and let $z_i' = f_i(z_1, z_2, \ldots, z_n)$ $i = 1, 2, \ldots, n$ be the covering function assignment defined on the set S[u]. Then if the functions are considered as describing a circuit starting in the initial state u, the circuit is distributive with $D{:}C[u] = \Lambda$.

<u>Proof:</u> Let $\underline{a}$ and $\underline{b}$ be any two vectors in Λ such that $\underline{a}$ covers $\underline{b}$. Let a be the image of $\underline{a}$ and b be the image of $\underline{b}$ in S[u]. Since $\underline{a}$ covers $\underline{b}$ we have for one node i, $\underline{a}_i = \underline{b}_i + 1$ hence by (12:14) $z_i' = a_i$. Furthermore by (12:12) $b_j = a_j$ for $j \neq i$ hence $b \mathcal{R} a$. Therefore if $\underline{0}, \underline{a}(1), \underline{a}(2), \ldots, \underline{a}(k)$ is a covering sequence in Λ and $u, a(1), a(2), \ldots, a(k)$ its image in S[u], then $u \mathcal{R} a(1) \mathcal{R} a(2) \mathcal{R} \ldots \mathcal{R} a(k)$. Thus the functions $z_i' = f_i(z_1, z_2, \ldots, z_n)$ generate every covering sequence in the lattice.

Now let a and be be any two states in S[u] satisfying $a \mathcal{R} b$. Since a is in S[u], a is the image of at least one vector $\underline{a}$ in Λ. If $b_i \neq a_i$ then by (12:14) $z_i' = b_i$ and either $b_i = a_i + 1$ or $b_i = a_i - 1$. Thus there exists in Λ the state $\underline{b} = \underline{a} + L[a,b]$ where $\underline{a} \mathcal{R} \underline{b}$. Therefore any sequence $u \mathcal{R} a(1) \mathcal{R} a(2) \mathcal{R} \ldots \mathcal{R} a(k)$ in S[u] has the image $\underline{0} \mathcal{R} \underline{a}(1) \mathcal{R} \underline{a}(2) \mathcal{R} \ldots \mathcal{R} \underline{a}(k)$ in Λ where $\underline{a}(i) = L[u, a(1), a(2), \ldots, a(i)]$. Using (7:9) we infer that the lattice of states generated by the functions $z_i' = f_i(z_1, z_2, \ldots, z_n)$ is semimodular.

Furthermore since $C[u] = \Lambda$ we have that the lattice is distributive. Finally by (11:B) and (12:5) we obtain the equality of the change chart Σ and the set of causation signals $\Sigma[u]$.

12.10 We have shown so far that given a consistent state assignment $S[u]$ the covering functions generate the lattice Λ. We now would like to show the sufficiency of conditions (11:E) and (11:C) where $\Sigma[u]$ is replaced by the change chart Σ. We note that if Σ is finite then condition (11:5) trivially obtains. Furthermore Λ is also finite and if we define $S[u]$ by letting $u + \underline{a} = a$ be the image of $\underline{a}$ in Λ, then $S[u]$ is an isomorphic image of Λ. Therefore if $\underline{a}$ and $\underline{b}$ map onto the same state b in $S[u]$ we must have $\underline{a} = \underline{b}$ and hence $z'(\underline{a}) = z'(\underline{b})$. Thus, this particular state assignment is consistent and functions exist which will generate Λ. Consequently we turn our attention to the case where Σ is infinite. Here we will give an example showing that conditions (11:E) and (11:C) are not sufficient to obtain a consistent state assignment. Before we give this example we wish to point out that if node i has the property that $[\alpha, i]$ is in Σ for all α, then in any state assignment node i can never be either monotonely increasing or decreasing. This is inferred immediately from the fact that $S[u]$ is finite. Thus for some $\underline{a} \sim [\alpha, i]$, $\underline{b} \sim [\alpha + 1, i]$ and $\underline{c} \sim [\alpha + 2, i]$ we will have $a_i + 1 = b_i = c_i + 1$, that is b_i is a relative maximum.

Consider now the change chart in Figure 8. It is easily verified that this chart satisfies both (11:C) and (11:E). Moreover it could be a chart for a binary counter for every change on node number two implies two changes occuring at node number one. If we take into consideration node number one, we discover that there are only two possible cases for the attainment of a relative maximum. These are $\underline{a} \sim [2k + 1, 1]$ for some k is a relative maximum or $\underline{a} \sim [2k + 2, 1]$ for some k is a relative maximum. The lattice for these states is given in Figure 9. If now for the vector $(2k + 1, k)$ node number one attained a relative

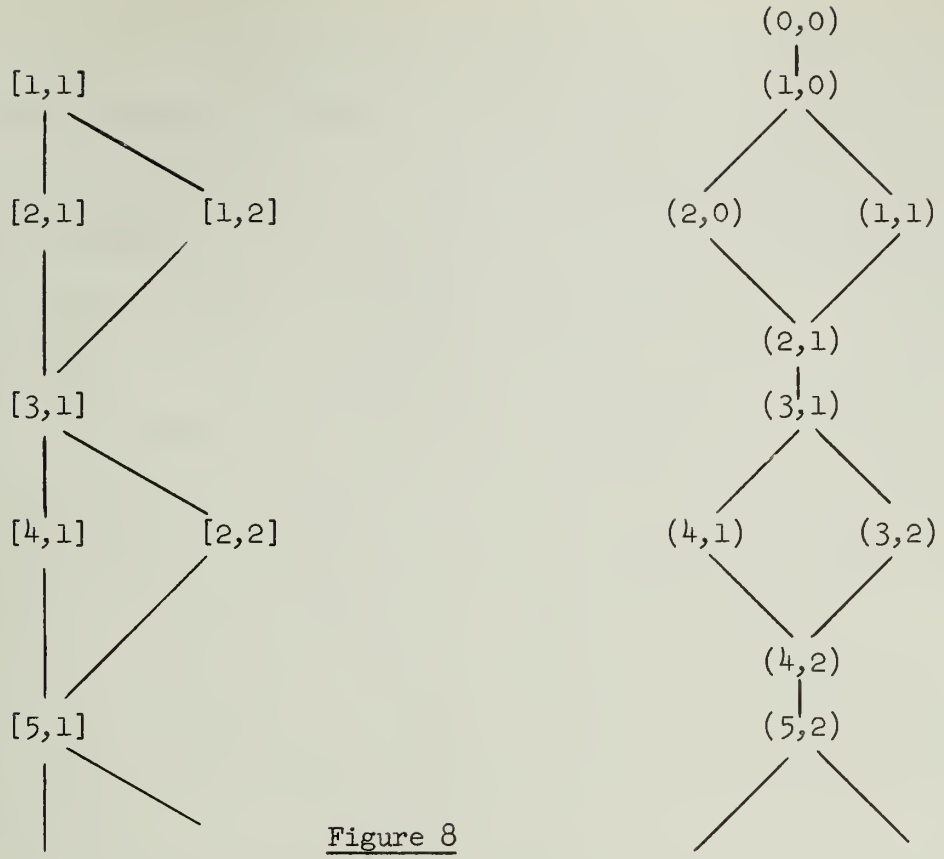

Figure 8

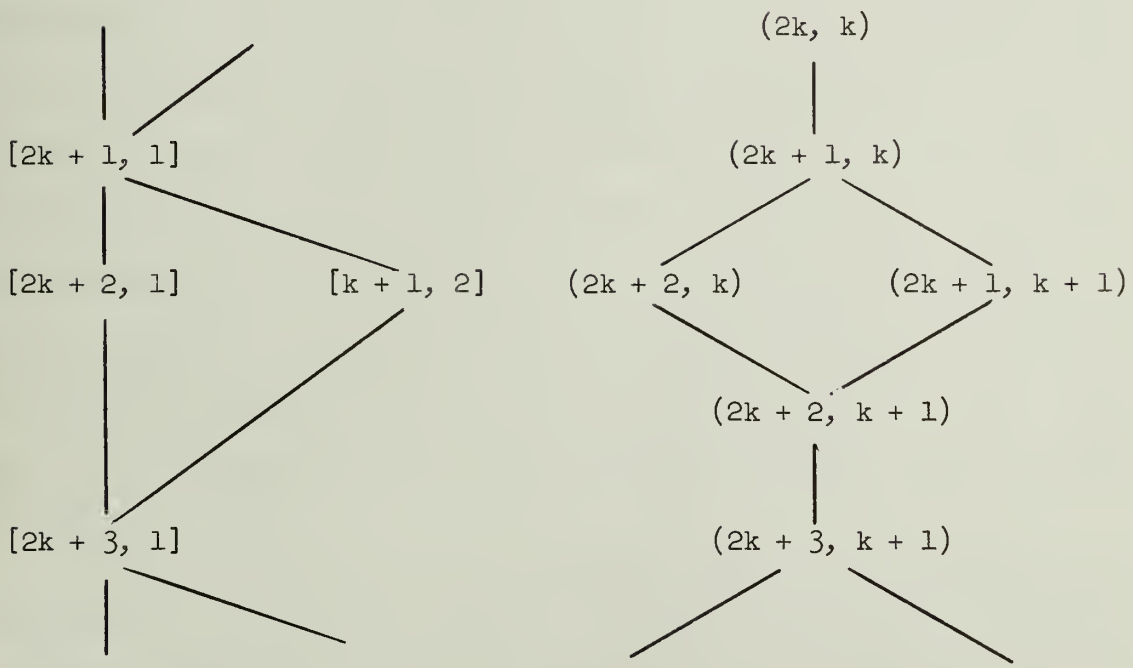

Figure 9

maximum, we would then have that both (2k, k) and (2k + 2, k) mapped onto the same state in the state assignment. However for the state (2k, k) node number two is in equilibrium and for the state (2k + 2, k) node number two is excited, thus yielding an inconsistent assignment. On the other hand if (2k + 2, k) corresponded to a relative maximum of node number one, then the vectors (2k + 1, k + 1) and (2k + 3,k + 1) would map onto the same state in the state assignment. But also for these vectors, node number two is in equilibrium in the former and excited in the latter. Thus in either case we cannot obtain a consistent assignment, hence conditions (11:C) and (11:E) are not sufficient to obtain a consistent assignment of states. This last example leads to the following theorem about distributive circuits.

(12:E) If there exists $[\gamma, i]$ in $\Sigma[u]$ such that for every $\alpha \geq \gamma$, $[\alpha + 1, i]$ is in $\Sigma[u]$ and covers $[\alpha, i]$, then the vector w which spans node i only spans node i.

<u>Proof:</u> We note that i is one of the i(s) in theorem (11:E) hence we first choose an $\alpha \geq \alpha(s)$ where i is taken to be i(s). Let $[\beta(k), i(k)]$ $k = 1, 2, \ldots, r$ be chosen so that $k \neq s$ and $[\beta(k), i(k)]$ is covered either by $[\alpha + 2, i(s)]$ or $[\alpha + 1, i(s)]$ and let $\underline{b}(k) \sim [\beta(k), i(k)]$. Assume that $[\beta(k), i(k)$ is covered by $[\alpha + 2, i(s)]$. Then since i(k) is not i(s) and since $[\alpha + 2, i(s)]$ covers $[\alpha + 1, i(s)]$ we have $[\alpha + 1, i(s)] \not< [\beta(k), i(k)]$, hence $\underline{b}(k)_{i(s)} \leq \alpha$. On the other hand if $[\beta(k), i(k)]$, were covered by $[\alpha + 1, i(s)]$ a similar argument would yield $\underline{b}(k)_{i(s)} \leq \alpha - 1$. In either case $\underline{b}(k)_{i(s)} \leq \alpha$ for all k. Now let $\underline{a} \sim [\alpha, i(s)]$ and consider the state $\underline{b} = \underline{a} \cup \underline{b}(1) \cup \underline{b}(2) \cup \ldots \underline{b}(r)$. This state has $\underline{b}_{i(s)} = \alpha$ and $\underline{b}_{i(k)} \geq \beta(k)$, hence by (12:D) the $i(s)^{th}$ node is excited in state $\underline{b}$. Thus the state $\underline{c}$ exists having $\underline{c}_{i(s)} = \underline{b}_{i(s)} + 1$ and covering $\underline{b}$. Again applying (12:D) we see that this state possesses an excited i(s) node and the state $\underline{d}$ exists where $\underline{d}_{i(s)} = \underline{c}_{i(s)} + 1$ and covers $\underline{c}$. Thus we have been able

to find a covering sequence $\underline{b}$, $\underline{c}$, $\underline{d}$ where only the $i(s)^{th}$ node changes. Finally we note that since this condition holds for all $\alpha \geq \alpha(s)$, then by (12:3), some signal $[\alpha + 1, i(s)]$ corresponds to a relative maximum of the state value. In this case both $\underline{b}$ and $\underline{d}$ have the same terminal state, and since the $i(s)^{th}$ node is excited in $\underline{b}$, it follows that we can find a state $\underline{c}(1)$ covering $\underline{d}$ having the same terminal state as $\underline{c}$. Finally by induction we obtain the covering sequence $\underline{b}$, $\underline{c}$, $\underline{d}$, $\underline{c}(1)$, $\underline{d}(1)$, $\underline{c}(2)$, $\underline{d}(2)$,... etc. where only the $i(s)^{th}$ node changes and where $t(\underline{b}) = t(\underline{d}) = t(\underline{d}(k))$ and $t(\underline{c}) = t(\underline{c}(k))$.

Consider now any signal satisfying $[\alpha, i(s)] \leq [\beta, i(m)]$. For $i(m) \neq i(s)$ we would have $\underline{b}_{i(m)} < \beta$ and hence in the above covering sequence every state would have an $i(m)^{th}$ component $< \beta$, But by (11:E) some $[\gamma, i(s)]$ must exist for which $[\beta, i(m)] < [\gamma, i(s)]$. This along with the fact that for some k, $\underline{d}(k)_{i(s)} > \gamma$ yields a violation of (12:2b). Hence no such $[\beta, i(m)]$ exists from which we infer $w_{i(m)} = 0$ for that w which spans node $i(s)$.

12.11 The remaining development in this chapter will consist of showing the existence of a set of functions which generates a set of causation signals equal to a given change chart on a subset of nodes. The additional nodes are needed in some cases but not necessarily in all. However, the example in section 12.10 shows us there are cases where at least one node must be added before we can expect to find a set of functions for the circuit, that is "realizing" a given change chart. Thus given the freedom of adding nodes, we will demonstrate the realization of every change chart satisfying the conditions of theorem (11:E). Hence we define:

(12:17) A change chart Σ is said to be realizable if and only if for every $i(j)$, $j = 1, 2, \ldots, k$ satisfying $[\gamma, i(j)]$ exists in Σ for all γ, there exists a signal $[\alpha(j), i(j)]$ and a positive integer $r(j)$ satisfying:

(12:17a) $[\alpha, i(s)] \leq [\beta, i(t)]$ if and only if $[\alpha + r(s), i(s)] \leq [\beta + r(t), i(t)]$ where $\alpha \geq \alpha(s)$

(12:17b) If $[\alpha(s), i(s)] \leq [\beta, i(t)]$ then there is some $[\gamma, i(s)]$ satisfying $[\beta, i(t)] \leq [\gamma, i(s)]$

(12:17c) If $[\alpha(s), i(s)] < [\beta, i(t)]$ and $[\beta, i(t)]$ covers $[\gamma, m]$ then there exists a $[\delta, m]$ satisfying $[\alpha(s), i(s)] \leq [\delta, m]$.

Except for (12:17c) this definition is precisely the same as the conditions of theorem (11:E). Moreover condition (12:17c) can be shown to follow from (12:17a) and (12:17b). For assume condition (12:17c) did not hold. Then there exists a maximum value of $[\beta, i(t)]$ for which (12:17c) fails. For if no such maximum existed then we would be able to find an infinite subset of signals each of which is covered by a $[\beta, i(t)]$ satisfying $[\alpha(s), i(s)] < [\beta, i(t)]$. But since the node index is restricted to a finite range it follows that in this set at least one of the nodes is an $i(j)$, say $i(q)$. Furthermore there is a signal in this set $[\gamma, i(q)]$ where $\gamma \geq \alpha(q)$. Let $[\beta, i(t)]$ cover $[\gamma, i(q)]$ then since $[\beta, i(t)] > [\alpha(s), i(s)]$ by (12:17b) there is a $[\delta, i(s)]$ satisfying $[\beta, i(t)] \leq [\delta, i(s)]$. Hence $[\gamma, i(q)] < [\delta, i(s)]$ and again by (12:17b) there is some $[\epsilon, i(q)]$ for which $[\delta, i(s)] \leq [\epsilon, i(q)]$. But this contradicts the condition that (12:17c) fails hence a maximum value of $[\beta, i(t)]$ exists and we may by (12:17b) choose an $[\alpha'(s), i(s)] \geq$ this maximum. Therefore (12:17) is completely equivalent to the conditions of theorem (11:E). The additional condition was added only to facilitate the later development.

In 11.8 we discussed the concept of a cycling set. These will play an important part in this development as the nodes in these sets determine the spanned nodes in the realization. However for our purposes we need only define the set $\{[\alpha(s), i(s)]\}$ rather than the complete cycling set.

(12:18) The set $\{[\alpha(s), i(s)]\}$ consists of every $[\beta, i(t)]$ satisfying $[\alpha(s), i(s)] \leq [\beta, i(t)]$ where $\alpha(s)$ is defined in (12:17).

We note firstly that since $[\gamma, i(s)]$ exists for all γ then the set $\{[\alpha(s), i(s)]\}$ is infinite. Secondly if $[\beta, m]$ is in $\{[\alpha(s), i(s)]\}$ then m is one of the $i(j)$. Hence if $[\beta, i(t)]$ is in $\{[\alpha(s), i(s)]\}$ where $i(s) \neq i(t)$ then there exists a set $\{[\alpha(t), i(t)]\}$ containing changes on the node $i(s)$. Thus although we may represent Σ as a finite part plus one or more sets of the form $\{[\alpha(s), i(s)]\}$ this representation is not unique.

12.12 We now introduce the concept of an isomorphism vector which will be shown to have properties very similar to a cycling vector. Indeed, given a causation signal set $\Sigma[u]$, an isomorphism vector would just be an integral multiple of a cycling vector.

(12:19) For each set $\{[\alpha(s), i(s)]\}$ and each set of isomorphism integers $r(j)$, there exists an n-dimensional isomorphism vector w satisfying:

(12:19a) $w_{i(t)} = r(t)$ if and only if $[\beta, i(t)]$ is in $\{[\alpha(s), i(s)]\}$ for some β.

(12:19b) $w_m = 0$ for all other components.

We note that if $w(1)$ and $w(2)$ are two different isomorphism vectors then $w(1) \wedge w(2)$ is the zero vector. This follows from the property that there exists a signal $[\beta, i(t)]$ lying in $\{[\alpha(s), i(s)]\}$ when and only when there exists a signal $[\gamma, i(s)]$ lying in $\{[\alpha(t), i(t)]\}$. Thus, like cycling vectors, $w(1)$ and $w(2)$ are orthogonal. The next property shows that the isomorphism vectors play a similar role in the lattice Λ as the cycling vectors in a distributive circuit.

(12:20) Let w be an isomorphism vector and let $\underline{a}$ be any vector in Λ satisfying $\underline{a}_{i(s)} \geq \alpha(s)$ where $w_{i(s)} \neq 0$ and $\alpha(s)$ is defined in (12:17). Then $\underline{a} + w$ is in Λ and is isomorphic to $\underline{a}$.

Proof: Condition (12:2a) applied to the vector $\underline{a} + w$ is immediately inferred from (12:17). To establish (12:2b) we note for any component i where $w_i = 0$ we have $(\underline{a} + w)_i = \underline{a}_i$. Hence (12:2b) needs only to be established for those components i(s) for which $w_{i(s)} \neq 0$. By (12:17c) if $[\beta, j] \leq [\underline{a}_{i(s)} + w_{i(s)}, i(s)]$ where $w_j = 0$ we must have $[\beta, j] \leq [\alpha(s), i(s)] \leq [\underline{a}_{i(s)}, i(s)]$. Thus applying (12:2b) to the vector $\underline{a}$, we obtain $\beta \leq \underline{a}_j$. But $w_j = 0$ hence $\underline{a}_j = \underline{a}_j + w_j$ and we therefore infer (12:2b) for this case. This then leaves us only the case where $[\beta, i(t)] \leq [\underline{a}_{i(s)} + w_{i(s)}, i(s)]$ and $w_{i(t)} \neq 0$. Denying (12:2b) for the i(t) component we have $\beta > (\underline{a} + w)_{i(t)}$. But since $w_{i(t)} \neq 0$ and $\underline{a}_{i(t)} \geq \alpha(t)$ we would have $\beta - w_{i(t)} \geq \alpha(t)$. Hence by (12:17) there is an isomorphic image of $[\beta, i(t)]$ satisfying $[\beta - w_{i(t)}, i(t)] \leq [\underline{a}_{i(s)}, i(s)]$. Furthermore $\beta - w_{i(t)} > \underline{a}_{i(t)}$ hence (12:2b) is violated for the vector $\underline{a}$. We therefore conclude, $\underline{a} + w$ is in Λ.

To show that $\underline{a} + w$ is isomorphic to $\underline{a}$ we need only show that for every vector $\underline{b}$ which covers $\underline{a} + w$ there is an isomorphic image $\underline{b} - w$ covering $\underline{a}$. Then since every vector $\underline{c}$ covering $\underline{a}$ has an image $\underline{c} + w$ covering $\underline{a} + w$, we have that $\underline{a}$ and $\underline{a} + w$ are isomorphic to each other. Let $\underline{b}$ cover $\underline{a} + w$ and let $\underline{b}_i = \underline{a}_i + w_i + 1$. We consider two cases $w_i = 0$ and $w_i \neq 0$. For the case $w_i = 0$ we have by (12:D), $\underline{a}_k + w_k \geq \beta$ for every $[\beta, k]$ covered by $[\underline{b}_i, i] = [\underline{a}_i + 1, i]$. If now $w_k = 0$ we have $\underline{a}_k \geq \beta$ thus the conditions of (12:D) hold for the kth node. Hence we need only consider the case where $w_k \neq 0$, that is $k = i(t)$ for some t. Here we must have $\beta < \alpha(t)$ for otherwise we would have $[\alpha(t), i(t)] \leq [\beta, i(t)] < [\underline{a}_i + 1, i]$ and hence $w_i \neq 0$ by (12:17). Therefore $\beta < \alpha(t)$ and since $\underline{a}_{i(t)} \geq \alpha$ we have $\underline{a}_{i(t)} \geq \beta$. Thus node i is excited in $\underline{a}$ by (12:D) and a state $\underline{c} = \underline{b} - w$ exists which covers $\underline{a}$.

We now consider the case where $w_i \neq 0$, hence $i = i(s)$ for some s. Since $\underline{a}_{i(s)} \geq \alpha(s)$ and since $\underline{b}_{i(s)} = \underline{a}_{i(s)} + w_{i(s)} + 1$ we have that there is an

isomorphic image of $[\underline{a}_{i(s)} + w_{i(s)} + 1, i(s)]$, namely $[\underline{a}_{i(s)} + 1, i(s)]$. Let $\underline{j}$ be the join-irreducible element corresponding to this signal and consider the state $\underline{a} \vee \underline{j} = \underline{c}$. This state has $i(s)^{th}$ component of $\underline{a}_{i(s)} + 1$. Assume for some other component k, $\underline{c}_k > \underline{a}_k$. Then $\underline{j}_k = \underline{c}_k$ and by (12:6) $[\underline{a}_k, k] < [\underline{c}_k, k] < [\underline{a}_{i(s)} + 1, i(s)]$. If $w_k = 0$ then since $[\underline{a}_k, k] < [\underline{c}_k, k] < [\underline{a}_{i(s)} + w_{i(s)} + 1, i(s)]$ we must have $\underline{b}_k \geq \underline{c}_k$ which implies $\underline{b}_k > \underline{a}_k + w_k$. But this contradicts the condition that $\underline{b}$ covers $\underline{a} + w$. On the other hand if $w_k \neq 0$ then since $\underline{a}_{i(t)} \geq \alpha(t)$ we have by (12:17) $[\underline{a}_{i(t)} + w_{i(t)}, i(t)] < [\underline{c}_{i(t)} + w_{i(t)}, i(t)] < [\underline{a}_{i(s)} + w_{i(s)} + 1, i(s)]$ where $k = i(t)$. Here too by (12:2b) we obtain $\underline{b}_{i(t)} > \underline{a}_{i(t)} + w_{i(t)}$ which contradicts $\underline{b}$ covering $\underline{a} + w$. Thus for no other component k is $\underline{c}_k > \underline{a}_k$, hence $\underline{c} = \underline{b} - w$ which completes the proof.

(12:20) shows there is an isomorphism carrying the set of all states $\underline{b} \geq \underline{a}$ onto the set of all states $\underline{c} \geq \underline{a} + w$ where $\underline{b}$ in the former set maps onto $\underline{b} + w$ in the latter. Moreover the isomorphism assures us that the same nodes are excited (in equilibrium) in $\underline{b} + w$ as in $\underline{b}$. Hence if we were able to choose a state assignment in such a way that if $\underline{a}$ and $\underline{b}$ mapped onto the same state, then $\underline{a}$ and $\underline{b}$ differed by multiples of isomorphism vectors, we would be able to obtain a consistent state assignment. The next and final result shows that this is always possible to do provided at least one new node is added for each isomorphism vector.

(12:F) If for every relizable change chart Σ one new node is added for each isomorphism vector, then there exists a set of functions which, when considered as a circuit starting in an initial state u, generates a distributive circuit D:C[u] possessing a set of causation signals $\Sigma[u]$ equal to Σ on the subset of nodes formed by eliminating the signals of the added nodes.

Proof: The proof will be carried out completely for the case where there is only one isomorphism vector. In this case we will add a new node and thus define a new change chart which will still be realizable and which will possess a new isomorphism vector related to the isomorphism vector in the old chart. Then we will define a state assignment for the lattice Λ which satisfies the property that $\underline{a}$ and $\underline{b}$ both map onto the same state a if and only if $\underline{a}$ and $\underline{b}$ differ by an even multiple of the new isomorphism vector. Since $\underline{a}$ and $\underline{b}$ will then be isomorphic states in the lattice it follows that the state assignment is consistent, hence functions exist which will generate the modified chart. Further since the new chart has the same ordering relations as the original chart on their common nodes, the proof is established for the case of one isomorphism vector. But this vector is arbitrarily chosen hence the proof is trivially extended to the general case.

Without loss of generality we may let 1, 2,..., k be those nodes for which $[\alpha, i]$ exists for all α. Let $[\alpha^*(j), j]$ $j = 1, 2, \ldots, k$ be chosen so that $[\alpha^*(j), j]$ is in $\{[\alpha(1), 1]\}$ and $[\alpha^*(j) - 1, j]$ is not in $\{[\alpha(1), 1]\}$ when $[\alpha^*(j) - 1, j]$ exists, $\{[\alpha(1), 1]\}$ being defined by (12:18). Choose a set of isomorphism integers $r^*(j)$ so that $\{[\alpha(1), 1]\}$ and $\{[\alpha(1) + r^*(1), 1]\}$ are isomorphic and $[\alpha^*(j), j] \leq [\alpha(1) + r^*(1), 1]$ for all j. This may be done by virtue of condition (12:17b). Let w^* denote the isomorphism vector for this set of isomorphism integers. We now form a new change chart from the old by adding changes corresponding to a new node. This node will be labeled as the 0th node and the changes are added as follows:

(12:21) $[\alpha^*(j) + (j - 1) w^*_j, j]$ is replaced by $[\alpha^*(j) + (j - 1) w^*_j, j]$ covered by $[j, 0]$ for $j = 1, 2, \ldots, k$. The sets $\{[\alpha^*(1), 1]\}$ and $\{[\alpha^*(1) + mkw^*_1, 1]\}$ are isomorphic for all m, the isomorphism integers being $r(0) = k$, $r(1) = kw^*_1, \ldots, r(j) = kw^*_j, \ldots r(k) = kw^*_k$.

Figure 10 depicts the new change chart, solid lines indicate $[\alpha,i] < [\beta,i]$ when in the downward direction. We note that since $[\alpha(m) + pw^*_m, m] < [\alpha(1) + (p + 1) w^*_1, 1]$ for $m \leq k$ and all p, then we have insured that $[q,0] < [q + 1, 0]$ always holds. Hence we have defined a new change chart, that is it satisfies (12:1). Furthermore by (12:21) it is still realizable. We now let $\beta(j) = \alpha^*(j) + (j - 1) w^*_j$ for $j = 1, 2,\ldots, k$ and $\beta(0) = 1$ for convenience in defining the state assignment.

For the state assignment we will take the initial state to be 0. This will not upset any of the analysis for the more general state assignment would be carried out by adding u to every state. Also we will not assign a state value to every component of a vector $\underline{a}$ in Λ but only to those components for which $w \neq 0$. In general the remaining components would be assigned as follows: those nodes possessing only a finite number of changes would be assigned as in the finite case discussed in Section 12.10. All other nodes would then have an infinite number of changes and would then be assigned in a similar fashion as the nodes 0, 1, 2,..., k now being discussed. Let $\underline{a}$ now, be any state in Λ, then the mapping of $\underline{a}$ onto the state a is defined as follows: Let $w(j) = kw^*_{(j)}$ for $j = 1, 2,\ldots, k$ and $w(0) = k$ then,

(12:22a) $\quad a_j = \underline{a}_j$ if $\underline{a}_j \leq \beta(j)$

(12:22b) $\quad a_j = \beta(j) + [\underline{a}_j - \beta(j)] \bmod w(j)$ if for some $m \geq 0$, $2mw(j) \leq \underline{a}_j - \beta(j) < (2m + 1) w(j)$.

(12:22c) $\quad a_j = \beta(j) + w(j) - [\underline{a}_j - \beta(j)] \bmod w(j)$ if for some $m \geq 0$, $(2m + 1) w(j) \leq \underline{a}_j - \beta(j) < (2m + 2) w(j)$.

Thus we see, [(12:22a)], that the jth node, $j = 1, 2,\ldots, k$, increases its state's value uniformly to $\beta(j)$. It then continues to increase, [(12:22b)], up to the value $\beta + w(j)$ where upon it decreases back to the value $\beta(j)$, [(12:22c)],

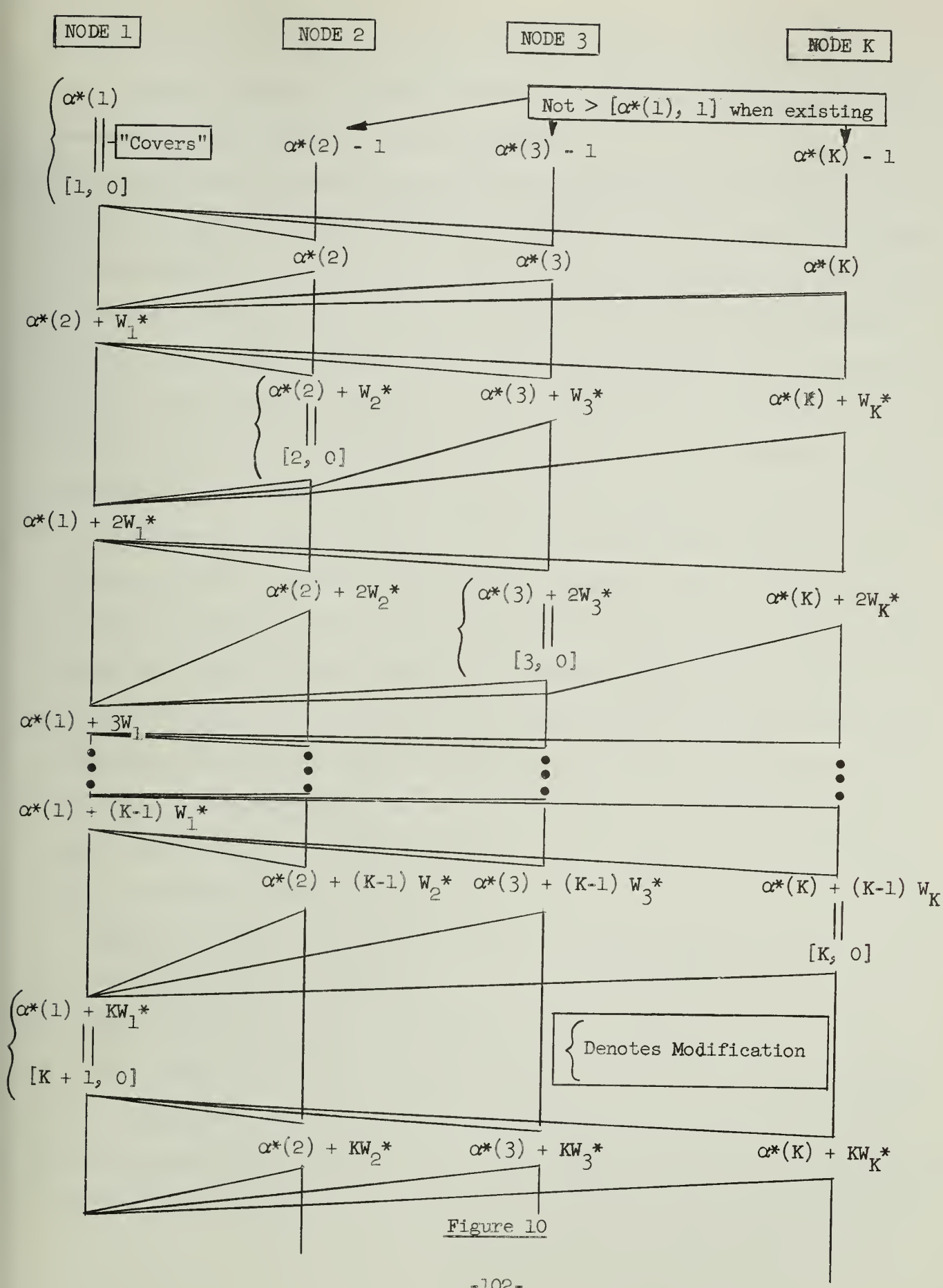
NODE 1
NODE 2
NODE 3
NODE K
Not > [α*(1), 1] when existing
α*(1)
"Covers"
[1, 0]
α*(2) - 1
α*(3) - 1
α*(K) - 1
α*(2)
α*(3)
α*(K)
α*(2) + W1*
α*(2) + W2*
[2, 0]
α*(3) + W3*
α*(K) + WK*
α*(1) + 2W1*
α*(2) + 2W2*
α*(3) + 2W3*
[3, 0]
α*(K) + 2WK*
α*(1) + 3W1
α*(1) + (K-1) W1*
α*(2) + (K-1) W2*
α*(3) + (K-1) W3*
α*(K) + (K-1) WK
[K, 0]
α*(1) + KW1*
[K + 1, 0]
Denotes Modification
α*(2) + KW2*
α*(3) + KW3*
α*(K) + KWK*

Figure 10

and then repeats. We shall call $\beta(j)$ and $\beta(j) + w(j)$ the minimum and maximum cyclic values. For the 0th node we see that it increases its value to $k + 1$ and decreases back to 1 before it starts repeating.

Consider now any two states $\underline{a}$ and $\underline{b}$ both mapping onto the same state a. By (12:12) we immediately infer that $\underline{a} \wedge \underline{b}$ also maps onto the same state a since its components are either $\underline{a}_i$ or $\underline{b}_i$. Thus we can without any loss of generality assume $\underline{a} < \underline{b}$. The theorem is now proved in three cases. The first case is $\underline{b}_0 \geq \underline{a}_0 + 2w(0)$. For this case we will show that $\underline{b}_j \geq \underline{a}_j + 2w(j)$. But this will imply $\underline{a}_j \geq \beta(j)$ hence by (12:20) $\underline{a} + 2w$ is isomorphic to $\underline{a}$ and satisfies $\underline{a} + 2w \leq \underline{b}$. Moreover $\underline{a} + 2w$ maps onto the same state as $\underline{a}$, hence we have reduced the problem to two further cases: namely $\underline{a}_0 = \underline{b}_0$ and $\underline{a}_0 < \underline{b}_0 < \underline{a}_0 + 2w(0)$. In the first of these we will obtain the trivial result $\underline{a} = \underline{b}$. The second of these cases is the most difficult and will be saved to the end of the theorem. Here we will show that $\underline{a}_0 + w(0) = \underline{b}_0$ hence in order for $a_0 = b_0$ we must have $w(0)$ even. At this point we must break the proof down further to two cases: namely $w(0) = 2$ and $w(0) \geq 4$. If $w(0) = 2$ we will show $a_2 \neq b_2$ which contradicts the condition that $\underline{a}$ and $\underline{b}$ map onto the same state. If however $w(0) \geq 4$ then we will show for some $m = 2, 3, \ldots, k$ $a_m \neq b_m$ which too is a contradiction. Thus the case $\underline{a}_0 < \underline{b}_0 < \underline{a}_0 + 2w(0)$ is impossible and we finally obtain the result that if $\underline{a}$ and $\underline{b}$ map onto the same state, then $\underline{a} + 2mw = \underline{b}$ for some m.

<u>Case 1:</u> $\underline{a} < \underline{b}$ and both $\underline{a}$ and $\underline{b}$ map onto the same state a. Further $\underline{a}_0 + 2w(0) \leq \underline{b}_0$. Consider any node $j = 1, 2, \ldots, k$. By construction we have that there exist signals $[e(j), 0]$ and $[f(j), j]$ satisfying: $[\underline{a}_0, 0] < [f(j), j] \mathcal{C} [e(j), 0] \leq [\underline{a}_0 + w(0), 0]$ where $[f(j), j]$ corresponds to either a maximum or minimum cyclic value of node j and $\mathcal{C}$ denotes "covered by". However the sets $\{[\underline{a}_0, 0]\}$ and $\{[\underline{a}_0 + w(0), 0]\}$ are isomorphic hence $[\underline{a}_0 + w(0), 0] < [f(j) + w(j), j] \mathcal{C} [e(j) + w(0), 0] \leq [\underline{a}_0 + 2w(0), 0]$. We note that $\underline{a}_j \leq f(i)$;

for if $\underline{a}_j > f(j)$ then we would have $[e(j), 0] < [f(j) + 1, j] \leq [\underline{a}_j, j]$, hence by (12:2b) $\underline{a}_0 \geq e(j)$ which is a contradiction. Also by hypothesis $[\underline{a}_0 + 2w(0), 0] \leq [\underline{b}_0, 0]$ hence by (12:2b) we must have $\underline{b}_j \geq f(j) + w(j)$. Since $[f(j), j]$ and $[f(j) + w(j), j]$ correspond to opposite extremal cyclic values of the state assignment and further since $\underline{a}_j \leq f(j)$ and $\underline{b}_j \geq f(j) + w(j)$, we must have $\underline{a}_j + 2w(j) \leq \underline{b}_j$. Now $\underline{a}_j$ cannot be $< \beta(j)$ for if it were then $\underline{a}$ and $\underline{b}$ mapping onto the same state would imply $\underline{a}_j = \underline{b}_j$ which is a contradiction. Thus we may apply (12:20) to the vector $\underline{a}$ and form the vector $\underline{a} + 2w$. This vector is isomorphic to $\underline{a}$, maps onto the same state as $\underline{a}$, and, by the previous result $\underline{a} + 2w \leq \underline{b}$. This completes the first case. By finite induction we may now assume that either $\underline{a}_0 = \underline{b}_0$ or $\underline{a}_0 < \underline{b}_0 < \underline{a}_0 + 2w(0)$, where $\underline{a} < \underline{b}$ and both map onto the same state a.

<u>Case 2:</u> Here we assume $\underline{a}_0 = \underline{b}_0$ and we will show $\underline{a} = \underline{b}$. Assume for some node j, $\underline{a}_j < \underline{b}_j$. Then an extremal cyclic value of node j must occur somewhere in the interval $[\underline{a}_j, j] < [\underline{b}_j, j]$, say $[c,j]$. Moreover we may choose this signal so that $\underline{a}_j < c < \underline{b}_j$. This is evident for the case where neither $\underline{a}_j$ nor $\underline{b}_j$ mapped onto an extremal value. For the case where $\underline{a}_j$ did map onto an extremal value then $\underline{b}_j$ must map onto the same extremal value hence $[c,j]$ would correspond to the opposite extremal value. Let $[d,0]$ be that signal on node 0 which covers $[c,j]$ then we have: $[\underline{a}_j, j] < [c,j] \mathcal{C} [d,0] < [\underline{b}_j, j]$. By (12:2b) we have $\underline{b}_0 \geq d$. But also by (12:2b) $\underline{a}_0 < d$ for if $\underline{a}_0 \geq d$ then $\underline{a}_j$ must satisfy $\underline{a}_j \geq c$ which it cannot. Hence $\underline{a}_0 \neq \underline{b}_0$ which contradicts the original assumption. Hence we obtain the desired result $\underline{a}_j = \underline{b}_j$ for all $j = 0, 1, 2, \ldots, k$.

<u>Case 3:</u> Here $\underline{a}_0 < \underline{b}_0 < \underline{a}_0 + 2w(0)$. Since $\underline{a}_0$ and $\underline{b}_0$ map onto the same state value, node 0 must attain an extremal value corresponding to some $[m,0]$ where $[\underline{a}_0, 0] < [m,0] < [\underline{b}_0, 0]$. But by construction for every node $j = 1, 2, \ldots, k$

we have that some $[c(j), j]$ exists such that $[m,0] < [c(j), j] < [m + 1, 0]$. Hence $[\underline{a}_0, 0] < [m,0] < [c(j), j] < [\underline{b}_0, 0]$. By (12:2b) we must have $\underline{b}_j \geq c(j)$. Also $\underline{a}_j$ must be $< c(j)$ for otherwise (12:2b) would imply that $\underline{a}_0 \geq m$ which is a contradiction. Therefore $\underline{a}_j < \underline{b}_j$ for each $j = 1, 2, \ldots, k$, and there exists an extremal value corresponding to some $[d(j), j]$ satisfying $[\underline{a}_j, j] < [d(j), j] < [\underline{b}_j, j]$. Since $[d(j), j]$ corresponds to an extremal value it is covered by a change on node 0, say $[e(j), 0]$. Hence we may write $[\underline{a}_j, j] < [d(j), j] \mathcal{C} [e(j), 0] < [\underline{b}_j, j]$. Now by (12:2b) we obtain $\underline{a}_0 < e(j)$ and $\underline{b}_0 \geq e(j)$. But since there are k distinct values of $e(j)$ we obtain the condition $\underline{a}_0 + k \leq \underline{b}_0$.

We have previously shown that $\underline{a}_j < \underline{b}_j$ for $j = 1, 2, \ldots, k$. Hence in order that both $\underline{a}$ and $\underline{b}$ map onto the same state we must have $\underline{a}_j \geq \beta(j)$. Thus we may apply (12:20) to the vector $\underline{a}$ and obtain the vector $\underline{a} + 2w$. This vector maps onto the same state as $\underline{a}$ and is isomorphic to $\underline{a}$. Moreover $\underline{b}_0 < \underline{a}_0 + 2w(0) < \underline{b}_0 + 2w(0)$. Therefore if we substitute $\underline{b}$ for $\underline{a}$ and $\underline{a} + 2w$ for $\underline{b}$ in the previous argument we would obtain $\underline{a}_0 + 2k \geq \underline{b}_0 + k$. This combined with the previous results yields $\underline{a}_0 + k = \underline{b}_0$. Now since only an even number of changes between $\underline{a}_0$ and $\underline{b}_0$ will insure that they map onto the same state we must have k even. There now remain but two cases to examine, namely $k = 2$ and $k \geq 4$. These will be called 3a, and 3b respectively. But before we examine them we need a small preliminary development.

Let $[\beta,m]$ be that signal covered by $[\underline{a}_0, 0]$ which corresponds to an extremal value on node m. Then since $\underline{b}_0 = \underline{a}_0 + k$ we have $[\underline{b}_0, 0]$ covers $[\beta + w(m), m]$ and $[\beta + w(m)]$ corresponds to the other extremal value. Also $m \neq 1$ for if it were then by construction $[\underline{a}_0, 0]$ would correspond to one extremal value of node 0 and $[\underline{b}_0, 0]$ would correspond to the other, which is a contradiction.

Case 3a: Here $k = 2$ (only two nodes in the diagram) and $m = 2$. A diagram of the chart is shown in Figure 11.

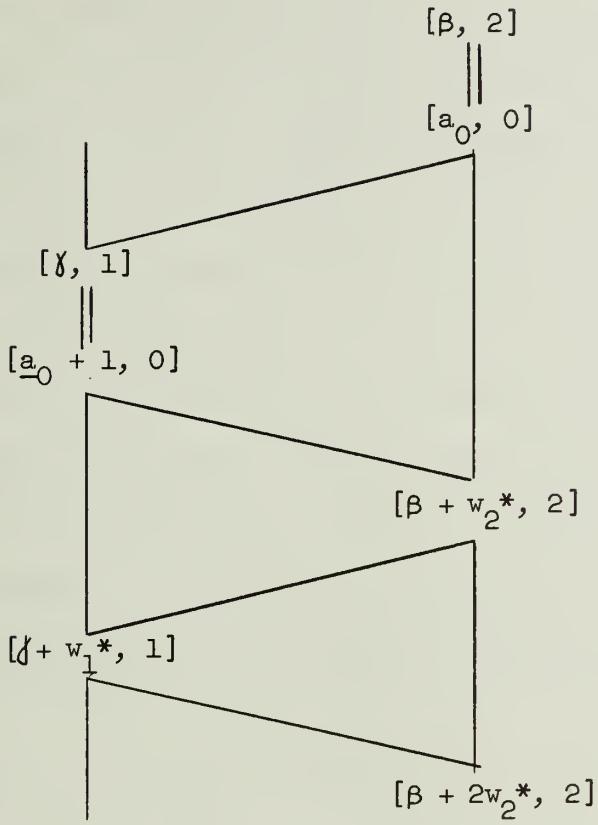

Figure 11

From the diagram we have $[\beta, 2] \mathcal{C} [\underline{a}_0, 0] < [\underline{a}_0 + 1, 0] < [\beta + w_2^*, 2]$. Hence by (12:2b) we obtain $\beta \leq \underline{a}_2 < \beta + w_2^*$. By (12:17) and the fact that $\underline{a}_0 + k = \underline{b}_0$ we have $[\beta + 2w_2^*, 2] \mathcal{C} [\underline{b}_0, 0] < [\underline{b}_0 + 1, 0] < [\beta + 3w_2^*, 2]$, hence (12:2b) implies $\beta + 2w_2^* \leq \underline{b}_2 < \beta + 3w_2^*$. If now $[\beta,2]$ corresponded to the minimum cyclic value, $\beta(2)$, then $[\beta + 2w_2^*, 2]$ would correspond to the maximum, $\beta(2) + 2w_2^*$ and we would have that the state assignments of $\underline{a}_2$ and $\underline{b}_2$ must satisfy: $\beta(2) \leq a_2 < \beta(2) + w_2^*$

and $\beta(2) + 2w_2^* \leq b_2 < \beta(2) + w_2^*$. This, however, implies $a_2 \neq b_2$ which is a contradiction. One the other hand, had $[\beta,2]$ corresponded to a maximum cyclic value then the roles of $\underline{a}$ and $\underline{b}$ are interchanged and we would still obtain $a_2 \neq b_2$. Hence case 3a fails and we are left with:

<u>Case 3b:</u> $k \geq 4$. The diagram for this is shown in Figure 12 for the case $m \neq k$. If $m = k$ figure 11 may be referred to substituting m for node 2. Regardless of the value of m we see that $[\underline{a}_0 + 1, 0] < [\beta + 2w_m^*, m]$. Therefore an argument modeled after the case $k = 2$ would result in the two conditions: $\beta \leq \underline{a}_m \leq \beta + 2w_m^*$ and $\beta + kw_m^* \leq \underline{b}_m < \beta + (k + 2)\, w_m^*$. If now $[\beta,m]$ corresponded to the minimal cyclic value $\beta(m)$ we would then have that the state assignments a_m and b_m would satisfy: $\beta(m) \leq a_m < \beta(m) + 2w_m^*$ and $\beta(m) + kw_m^* \leq b_m < \beta(m) + (k - 2)\, w_m^*$. However $k \geq 4$ hence we obtain $a_m \neq b_m$ which is a contradiction. On the other hand had $[\beta,m]$ corresponded to a maximal cyclic value we still would have obtained $a_m \neq b_m$. Thus case 3 yields a contradiction and the theorem is now proved.

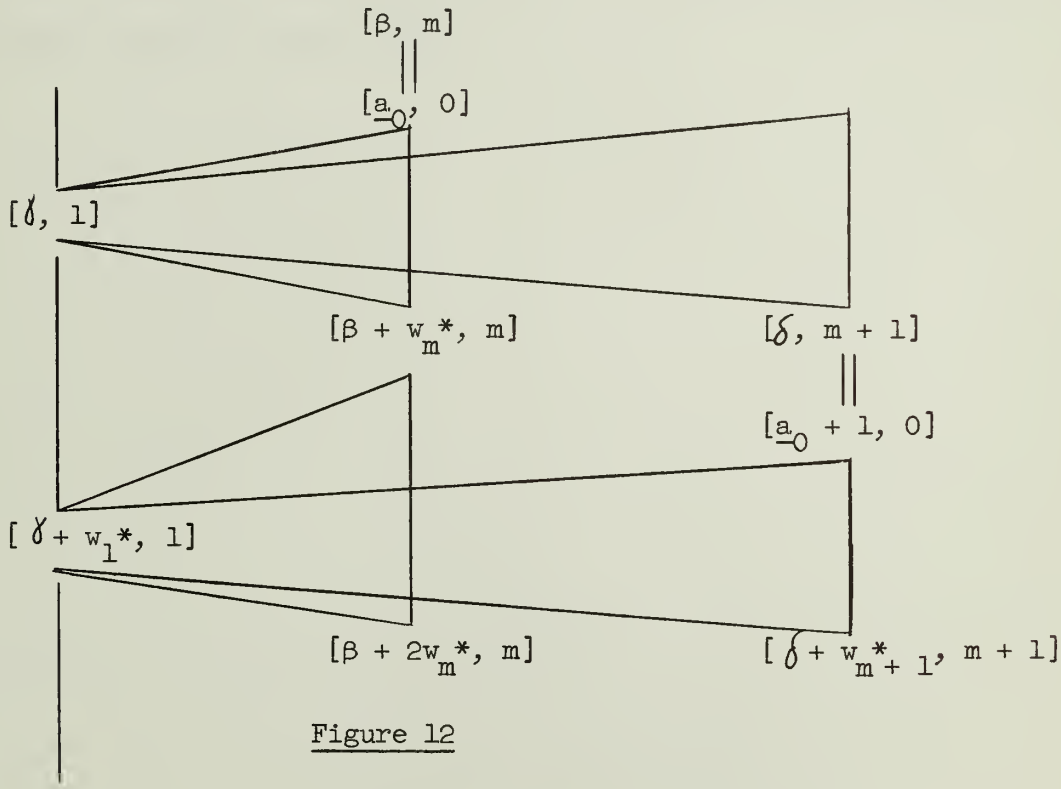

Figure 12

Theorem (12:F) concludes the development of this chapter. In Chapter 11 we have shown that given a distributive circuit we may find a set of causation signals $\Sigma[u]$ which satisfies the conditions of theorem (11:E). If we now consider this set as a change chart, then the initial development of this chapter shows us we are able to reproduce the lattice D:C[u]. Finally (12:F) shows that given a change chart Σ which satisfies the conditions of theorem (11:E) then by adding one node for each cycling set forming a new chart Σ', we are able to produce a distributive circuit possessing a set of causation signals $\Sigma[u]$ equal to Σ'. Thus we have been able to show that the conditions of theorem (11:E) are both necessary and sufficient to be able to find a set of functions which will generate a distributive circuit, provided we allow the freedom of adding new nodes to the change chart. The remaining development of this report will consist of examining the consequences of this synthesis procedure when we restrict the functions to being binary. We will show that provided we may add new nodes to the change chart, we are always able to find a binary distributive circuit which will synthesize the behavior of a given realizable change chart.

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